Algebra Comprehensive Exam Fall 2021, old style

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Answer five (5) questions only. You must answer *at least one* from each of section: (I) Groups, (II) Rings, and (III) Fields. Indicate CLEARLY which problems you want us to grade; otherwise, we will select the first problem from each section, and then the first two additional problems answered after that. Be sure to show enough work that your answers are adequately supported. Tip: When a question has multiple parts, the later parts often (but not always) make use of the earlier parts.

Notation: Unless otherwise stated, $\mathbb{N}$, $\mathbb{Q}, \mathbb{Z}, \mathbb{Z}_n, \mathbb{C}$, and $\mathbb{R}$ denote the sets of natural numbers, rational numbers, integers, integers modulo n, complex numbers, and real numbers respectively, regarded as groups or fields or vector spaces in the usual way.

Groups

(G1) Let G be an abelian group. Show that $H = \{x \in G \mid |x| \text{ is finite}\}$ is a subgroup of G. (Note: Here $|x|$ denotes the order of x.)

Solution: *Reminder: $|x|$ is finite if and only if $x^n = e$ for some n.*

(1) H closed under the group operation*: Let $x, y \in H$. Then $x^m = y^n = e$ for some $m, n \in \mathbb{Z}$, so using associativity and commutativity, $(xy)^{mn} = (x^m)^n (y^n)^m = e^n e^m = e$, hence $xy \in H$.*

(2) $e \in H$*: Obvious, since $|e| = 1$.*

(3) H closed under taking inverses*: Let $x \in H$. Then $x^n = e$ for some n and so $x^{-1} = x^{n-1}$. Hence $(x^{-1})^n = (x^{n-1})^n = x^{n(n-1)} = (x^n)^{n-1} = e^{n-1} = e$ and $x^{-1} \in H$.*

OR

Since $\langle x \rangle = \langle x^{-1} \rangle$, we have $|x^{-1}| = |x|$, from which the claim is clear.

(G2) The **center** of a group G is defined as

$$Z(G) = \{g \in G : gx = xg \text{ for all } x \in G\}.$$

(a) Prove $Z(G)$ is a normal subgroup of G.
(b) Prove: If $G/Z(G)$ is cyclic, then G is abelian.

Solution: *For problem (a) see here: https://en.wikipedia.org/wiki/Center_(group_theory) and also problem 2 from here: http://pi.math.cornell.edu/~riley/Teaching/Groups_and_Geometry2012/past_exams/2011prelim2_with_solutions.pdf. For (b) see problem 11 from here on page 2: https://www.math.utah.edu/~schwede/math435/HW4Sols.pdf*

(G3) *Let G be a group and $k \in \mathbb{N}$. Prove: If H is the only subgroup of G with order k, then H is a normal subgroup of G.*

Solution: *Suppose that H is the only subgroup of G with order k. Let $g \in G$. Define $\phi_g : G \to G$ by $\phi_g(x) = g^{-1}xg$. First show that ϕ_g is an isomorphism. Then $\phi_g(H) = g^{-1}Hg$ will be a subgroup of G of the same size as H. Thus, $g^{-1}Hg = H$. Since this is true for all $g \in G$ we have that H is a normal subgroup of G.*

Rings

(R1) *Prove that $2\mathbb{Z}$ is not isomorphic to $3\mathbb{Z}$ as rings.*

Solution: *Suppose $\phi : 2\mathbb{Z} \to 3\mathbb{Z}$ is ring homomorphism. Let $x = \phi(2)$. Then $2x = \phi(2) + \phi(2) = \phi(4) = \phi(2)\phi(2) = \phi(2)^2 = x^2$. Thus, $x^2 - 2x = 0$. So, $x(x-2) = 0$. Thus $x = 0$ or $x = 2$. Since $2 \notin 3\mathbb{Z}$ we have that $x = 0$. Thus $\phi(2) = 0$. It follows that $\phi(2k) = \phi(2)\phi(k) = 0$ for all k. So ϕ must be the zero map. And hence ϕ cannot be an isomorphism.*

- OR -

$2\mathbb{Z}$ and $3\mathbb{Z}$ are both infinite cyclic groups under addition, so are isomorphic groups. Their generators are $\{-2, 2\}$ and $\{-3, 3\}$ respectively. Any ring isomorphism $\phi : 2\mathbb{Z} \to 3\mathbb{Z}$ is, in particular, an abelian group isomorphism so must map generators to generators. Thus $\phi(2) = 3$ or $\phi(2) = -3$. But then $\phi(4) = \phi(2+2) = \phi(2) + \phi(2) = \pm(3+3) = \pm 6$ and $\phi(4) = \phi(2 \cdot 2) = \phi(2)\phi(2) = 3 \cdot 3 = 9$, an obvious contradiction.

(R2) *Let $R = \mathbb{Q}[x]$, the ring of polynomials over $\mathbb{Q}$. Let $I = (x^2 - 1, x^3 - 1)$, the ideal generated by $x^2 - 1$ and $x^3 - 1$. Is R/I a field? Explain.*

Solution: *By the Euclidean Algorithm, $\gcd[x^2 - 1, x^3 - 1] = x - 1$, so $I = (x-1)$ and $R/I = R/(x-1)$. Since $x - 1$ is irreducible, R/I is a field.*

- OR -

Let $\phi : R \to \mathbb{Q}$ be the evaluation homomorphism defined by $\phi(f) = f(1)$ for all $f \in R$. Clearly, $\operatorname{im}\phi = \mathbb{Q}$. Since $\phi(x^2 - 1) = \phi(x^3 - 1) = 0$, we have $I \subseteq \ker\phi$. To prove the opposite inclusion we first note that $x - 1 = (x^3 - 1) - x(x^2 - 1)$ and so $x - 1 \in I$.
Now suppose that $f \in \ker\phi$. Then $f(1) = 0$ and by the division algorithm, $f(x) = (x-1)g(x)$ for some $g \in R$. Since $x - 1 \in I$ we have $f \in I$. This shows that $I = \ker\phi$ and so $R/I = R/\ker\phi \cong \operatorname{im}\phi = \mathbb{Q}$, which is a field.

(R3) *Let $\{I_i \mid i \in \mathbb{N}\}$ be a set of ideals in a commutative ring R such that $I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$. Show that $I = \bigcup_i I_i$ is an ideal of R.*

Solution: *Let $r \in I$. Then $r \in I_i$ for some $i \in \mathbb{N}$. If $a \in R$, then $ar \in I_i \subseteq I$. Therefore I is closed under multiplication by elements of R.*
Let $r, s \in I$. Then $r \in I_i$ and $s \in I_j$ for some $i, j \in \mathbb{N}$. Because of the nesting of the ideals, r and s are both in $I_{\max(i,j)}$, and so $r + s \in I_{\max(i,j)} \subseteq I$. Thus I is closed under addition. Therefore I is an ideal of R.

Fields

(F1) *Let $\phi : F \to R$ be a nontrivial (i.e. nonzero) ring homomorphism with F a field and R a ring. Show that ϕ is injective (one-to-one).*

Solution: *Suppose that ϕ is not injective. Then $\ker \phi$ is nonzero and there is a nonzero element $r \in F$ such that $\phi(r) = 0$. Since F is a field, $r^{-1} \in F$ exists and $rr^{-1} = 1$. Hence*

$$\phi(1) = \phi(rr^{-1}) = \phi(r)\phi(r^{-1}) = 0\phi(r^{-1}) = 0$$

But then, for any $s \in F$, we get

$$\phi(s) = \phi(1s) = \phi(1)\phi(s) = 0\phi(s) = 0$$

so ϕ is the trivial homomorphism.

(F2) *Let E be a field and $\phi : E \to E$ a nonzero ring homomorphism. Show that*

$$F = \{a \in E \mid \phi(a) = a\}$$

is a subfield of F.

Solution: *Since $\phi(1) = 1$ we have $1 \in F$. Let $a, b \in F$. Then*

$$\phi(a - b) = \phi(a) - \phi(b) = a - b$$
$$\phi(ab) = \phi(a)\phi(b) = ab$$

so $a + b, ab \in F$. So F is closed under addition, negation and multiplication. Suppose $a \in F$ is nonzero. Then a^{-1} exists in E and satisfies $aa^{-1} = 1$. Since

$$\phi(a^{-1}) = 1\phi(a^{-1}) = a^{-1}a\phi(a^{-1}) = a^{-1}\phi(a)\phi(a^{-1}) = a^{-1}\phi(aa^{-1}) = a^{-1}\phi(1) = a^{-1}1 = a^{-1}$$

we have $a^{-1} \in F$.
By various subfield tests, this shows that F is a subfield of E.

(F3) *Show that $\alpha = \sqrt{2} + \sqrt[3]{3}$ is irrational.*

Solution: *We have $(\alpha - \sqrt{2})^3 = 3$, that is, $\alpha^3 - 3\sqrt{2}\alpha^2 + 6\alpha - 2\sqrt{2} = 3$. Solving for $\sqrt{2}$ we get*

$$\sqrt{2} = \frac{6\alpha + \alpha^3}{2 + 3\alpha^2}$$

If α is rational, then this equation implies that $\sqrt{2}$ is rational, a contradiction.

-OR-

Squaring both sides of the above equation we get $\alpha^6 - 6\alpha^4 - 6\alpha^3 + 12\alpha^2 - 36\alpha + 1 = 0$, that is, α is a root of

$$x^6 - 6x^4 - 6x^3 + 12x^2 - 36x + 1.$$

By the Rational Roots Theorem, this polynomial has no rational roots.

-OR-

Since $\mathbb{Q}(\sqrt{2})$ is a degree 2 extension of $\mathbb{Q}$, all elements of $\mathbb{Q}(\sqrt{2})$ have degree 1 or 2 over $\mathbb{Q}$. If α is rational, then $\sqrt[3]{3} = \alpha - \sqrt{2} \in \mathbb{Q}(\sqrt{2})$. But, since $\sqrt[3]{3}$ has degree 3 over $\mathbb{Q}$, this is a contradiction.
