ALGEBRA COMPREHENSIVE EXAMINATION

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Directions: Answer 5 questions only. You must answer at least one from each of linear algebra, groups, and synthesis. Indicate CLEARLY which problems you want us to grade. Otherwise, we will select which ones to grade, and they may not be the ones that you want us to grade. Be sure to show enough work that your answers are adequately supported.

Notation: As usual, $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{Z}$, $\mathbb{Z}_n$, $\mathbb{C}$, and $\mathbb{R}$ denote the sets of natural numbers, rational numbers, integers, integers modulo n, complex numbers, and real numbers respectively, regarded as groups or fields or vector spaces in the usual way.

Linear Algebra

(L1) Let V be a vector space. Suppose $u, v, w \in V$ satisfy $2u + v = w$ and $u + 2v = w$. What does this tell you about the dimension of the subspace spanned by $\{u, v, w\}$?

`Answer:` *Subtracting the equations we get $(2u + v) - (u + 2v) = w - w$ which simplifies to $u - v = 0$, and so $u = v$. Plugging this back into either of the given equations gives $w = 3u = 3v$.*

So every vector in the span of $\{u, v, w\}$ is a scalar multiple of v and the dimension of the span of $\{u, v, w\}$ is 0 if $v = 0$ and 1 if v is nonzero.

(L2) Let V and W be vector spaces, and let $T\colon V \to W$ be a linear map. Let $v_1, \ldots, v_k \in V$.

(a) Assume $v_1, \ldots, v_k$ are linearly independent. Does it follow that $T(v_1), \ldots, T(v_k)$ are linearly independent? If yes, prove it. If no, find a condition on T that would allow us to conclude that $T(v_1), \ldots, T(v_k)$ are linearly independent, and prove it.

(b) Assume $T(v_1), \ldots, T(v_k)$ are linearly independent. Does it follow that $v_1, \ldots, v_k$ are linearly independent? If yes, prove it. If no, find a condition on T that would allow us to conclude that $v_1, \ldots, v_k$ are linearly independent, and prove it.

`Answer:`

(a) *It does not follow; but if T is injective then it does follow. If $a_1 T(v_1) + \cdots + a_k T(v_k) = 0$, then $T(a_1v_1 + \cdots + a_kv_k) = 0$; since T is injective, this implies $a_1v_1 + \cdots + a_kv_k = 0$. But $v_1, \ldots, v_k$ are linearly independent, so $a_i = 0$ for all i.*

(b) *Yes, it does follow. If $a_1v_1 + \cdots + a_kv_k = 0$, then applying T to both sides yields $a_1 T(v_1) + \cdots + a_k T(v_k) = 0$. But $T(v_1), \ldots, T(v_k)$ are linearly independent, so $a_i = 0$ for all i.*

(L3) Let V be an n-dimensional vector space, and let $T\colon V \to V$ be an invertible linear operator. Let $v \in V$ be an eigenvector of T with eigenvalue λ.

(a) Prove that $\lambda \neq 0$.

(b) Prove that v is also an eigenvector of T^{-1}. What is the eigenvalue of T^{-1} corresponding to v?

`Answer:`

(a) *If $\lambda = 0$, then $T(v) = 0$, so T is not injective and hence not invertible.*

(b)

$$\begin{aligned}
\lambda v &= T(v) \\
T^{-1}(\lambda(v)) &= T^{-1}(T(v)) \\
\lambda T^{-1}(v) &= v \\
T^{-1}(v) &= \frac{1}{\lambda} v
\end{aligned}$$

Therefore v is an eigenvector of T^{-1}, and its eigenvalue is $1/\lambda$.

Groups

(G1) Let G be a group. Show that the following conditions are equivalent:

(a) G has exactly two subgroups.

(b) $G \cong \mathbb{Z}_p$ for some prime number p.

`Answer:` *Suppose that G has exactly two subgroups. Then G has at least 2 elements, so there is an element $a \in G$ which is not the identity element e. The subgroup $\langle a \rangle$ contains two different elements, e and a, so is not the trivial subgroup. By hypothesis, $\langle a \rangle$ must be all of G. That is, $G = \langle a \rangle$, G is a cyclic group and $G \cong \mathbb{Z}_n$ for some $n \geq 1$ or $G \cong \mathbb{Z}$. But we know that $\mathbb{Z}$ has infinitely many subgroups, and we know that $\mathbb{Z}_n$ has as many subgroups as n has (positive) divisors. Since the number of subgroups is a structural (algebraic) property, we must have $G \cong \mathbb{Z}_n$ with n having exactly two divisors. But this means that n is prime.*

Conversely, if $G \cong \mathbb{Z}_p$ with p prime, then G has the same number of subgroups as $\mathbb{Z}_p$, which is the same as the number of (positive) divisors of p, namely 2.

(G2) Let H be a subgroup of S_n for some $n \geq 2$. Show that, either all elements of H are even, or exactly half are even and half are odd.

`Answer:` *If all elements of H are even then we are done. Otherwise H contains an odd element σ. Consider the map $\lambda : H \to H$ defined by $\lambda(h) = \sigma h$ for all $h \in H$. Then λ is a bijection, and it interchanges the set of even elements of H with the set of odd elements of H. Thus H has exactly as many even elements as odd elements, and we are done.*

(G3) Consider a group G acting on a non-empty set X. We say the group action is **transitive** if, for every $x, y \in X$, there exists $g \in G$ such that $g \cdot x = y$. Given $g \in G$ and $x \in X$, x is a **fixed point** of g if $g \cdot x = x$. We say that the group action is **free** if the identity element of G is the only group element with a fixed point (that is, if $g \cdot x = x$ for some x, then g is the identity element of G). Prove that, if the action of G on X is both transitive and free, then $|G| = |X|$.

`Answer:` *Fix $x_0 \in X$. Define $\varphi\colon G \to X$ by $\varphi(g) = g \cdot x_0$. We will prove that φ is a bijection, which will imply that $|G| = |X|$.*

- *If $\varphi(g_1) = \varphi(g_2)$, then $g_1 \cdot x_0 = g_2 \cdot x_0$, so ${g_2}^{-1} g_1 \cdot x_0 = x_0$. Since the action is free, this implies that ${g_2}^{-1} g_1 = 1$, so $g_1 = g_2$. Therefore φ is injective.*
- *Let $x \in X$. Since the action is transitive, there exists $g \in G$ such that $g \cdot x_0 = x$. So $\varphi(g) = x$. Therefore φ is surjective.*

Synthesis

(S1) Let V be a vector space over a field F. Let $G = \{T : V \to V \mid T \text{ is a linear transformation}\}$.

(a) Show that G is a group under function addition. That is, the group operation is defined to be $(T_1 + T_2)(x) = T_1(x) + T_2(x)$ when $T_1, T_2 \in G$.

`Answer:` *The identity element is the zero function $T_0(x) = 0$ for all $x \in V$. Let $T_1, T_2, T_3 \in G$. Given $x \in V$ we have that*

$$((T_1 + T_2) + T_3)(x) = (T_1(x) + T_2(x)) + T_3(x)$$

$$= T_1(x) + (T_2(x) + T_3(x)) = (T_1 + (T_2 + T_3))(x)$$

which gives associativity. We have that $T_1 + T_2 \in G$ since

$$(T_1 + T_2)(\alpha x_1 + \beta x_2) = T_1(\alpha x_1 + \beta x_2) + T_2(\alpha x_1 + \beta x_2)$$

$$= \alpha T_1(x_1) + \beta T_1(x_2) + \alpha T_2(x_1) + \beta T_2(x_2)$$

$$= \alpha(T_1 + T_2)(x_1) + \beta(T_1 + T_2)(x_2).$$

So G is closed under function addition. Also, $-T_1$ is also in G since

$$-T_1(\alpha x_1 + \beta x_2) = -T_1(\alpha x_1) - T_1(\beta x_2) = -\alpha T_1(x_1) - \beta T_1(x_2)$$

$$= \alpha(-T_1)(x_1) + \beta(-T_1)(x_2).$$

So G is closed under inversion. Thus G is a group.

(b) Let

$$H = \{T \in G \mid \text{ there exists some } \alpha \in F \text{ such that } T(x) = \alpha x \text{ for all } x \in V\}$$

Show that H is a subgroup of G.

`Answer:` *First note that if $T : V \to V$ is defined by $T(x) = \alpha x$ then T is a linear transformation because*

$$T(c_1 v_1 + c_2 v_2) = \alpha(c_1 v_1 + c_2 v_2) = c_1 \alpha v_1 + c_2 \alpha v_2 = c_1 T(v_1) + c_2 T(v_2).$$

The zero function T_0 is in H since $T_0(x) = 0 \cdot x$ for all $x \in V$.

Let $T_1, T_2 \in H$. Then $T_1(x) = \alpha x$ and $T_2(x) = \beta x$ for all $x \in V$. Thus, $(T_1 + T_2)(x) = T_1(x) + T_2(x) = \alpha x + \beta x = (\alpha + \beta)x$. So $T_1 + T_2 \in H$. Also, $-T_1(x) = (-\alpha)x$ for all $x \in V$. Thus $-T_1 \in H$. So H is a subgroup of G.

(S2) Let $SO_2(\mathbb{R})$ be the group of 2×2 orthonormal matrices (that is, $A^{-1} = A^T$) with determinant 1 and real entries. Let $U(1) = \{z \in \mathbb{C} : |z| = 1\}$, considered as a group under multiplication. Prove that $SO_2(\mathbb{R})$ and $U(1)$ are isomorphic.

`Answer:` *Note that if $A \in SO_2(\mathbb{R})$, then $A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$, for some $a, b \in \mathbb{R}$ such that $a^2 + b^2 = 1$. Show that $\varphi : SO_2(\mathbb{R}) \longrightarrow U(1)$ given by*

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \longmapsto a + bi$$

is an isomorphism.

(S3) Let $GL_2(\mathbb{R})$ be the group of 2×2 invertible matrices over the real numbers, and $\mathbb{R}^*$, the group of nonzero real numbers under multiplication. Since the determinant function $\det : GL_2(\mathbb{R}) \to \mathbb{R}^*$ satisfies $\det(AB) = (\det A)(\det B)$ for all $A, B \in GL_2(\mathbb{R})$, it is a group homomorphism. Let G be a finite subgroup of $GL_2(\mathbb{R})$.

(a) Show that $\det(G) \leq \{1, -1\}$.

`Answer:` *The image of G is isomorphic to a quotient group of G so is a finite group. In particular, every element of $\det(G)$ must have finite order. But 1 and -1 are the only elements of $\mathbb{R}^*$ with finite order (they are the only solutions of equations like $x^n = 1$ with $n \in \mathbb{N}$). Thus $\det G \subseteq \{1, -1\}$. Since $\{1, -1\}$ is a subgroup of $\mathbb{R}^*$ we have $\det(G) \leq \{1, -1\}$.*

(b) Show that either, all matrices in G have determinant 1, or exactly half of the matrices in G have determinant 1.

`Answer:` *Either $\phi(G) = \{1\}$, the trivial subgroup, or $\phi(G) = \{1, -1\}$. In the first case, all matrices in G have determinant 1. In the second case, there are two cosets of $\ker\phi$, each containing the same number of matrices. All matrices in $\ker\phi$ have determinant 1. All other matrices in G have determinant -1.*