Algebra Comprehensive Exam Spring 2022, new style Solutions

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Answer five (5) questions only. You must answer *at least one* from each of section: (I) Linear algebra, (II) Group theory, and (III) Synthesis: linear algebra and group theory. Indicate CLEARLY which problems you want us to grade; otherwise, we will select the first problem from each section, and then the first two additional problems answered after that. Be sure to show enough work that your answers are adequately supported. Tip: When a question has multiple parts, the later parts often (but not always) make use of the earlier parts.

Notation: Unless otherwise stated, $\mathbb{Q}, \mathbb{Z}, \mathbb{Z}_n, \mathbb{C}$, and $\mathbb{R}$ denote the sets of rational numbers, integers, integers modulo n, complex numbers, and real numbers respectively, regarded as groups or fields or vector spaces in the usual way.

Linear algebra

(1) Let V be a vector space over the real numbers. Let W be the set of all functions from V to V.

(a) Prove that W is a vector space under pointwise addition and pointwise scalar multiplication. By "pointwise" we mean that if $f, g \in W$ and $k \in \mathbb{R}$, then we define $f + g\colon V \to V$ and $kf\colon V \to V$ by

$$(f + g)(v) = f(v) + g(v) \text{ and } (kf)(v) = k \cdot f(v).$$

(b) Let L be the set of all linear transformations from V to V. Prove that L is a subspace of W.

Solution:
(a) The proof at the website below works, with $S = V$.
https://yutsumura.com/vector-space-of-functions-from-a-set-to-a-vector-space/
(b) Certainly $L \subset W$.
Define $z\colon V \to V$ by $z(v) = 0$ for all $v \in V$. Then z is a linear transformation, and z is the zero element of W. So $z \in L$.
Suppose $f, g \in L$. We will show that $f + g \in L$. In other words, we will show that $f + g$ is a linear transformation from V to V. Let $v, w \in V$ and $k \in \mathbb{R}$. Then

$$\begin{aligned}(f+g)(v+w) &= f(v+w) + g(v+w) \text{ by def. of addition in } W\\ &= f(v) + f(w) + g(v) + g(w) \text{ because } f \text{ and } g \text{ are linear}\\ &= f(v) + g(v) + f(w) + g(w) \text{ because } V \text{ is a vector space}\\ &= (f+g)(v) + (f+g)(w) \text{ by def. of addition in } W\end{aligned}$$

Similarly, $(f + g)(kv) = k \cdot (f + g)(v)$.
Therefore $f + g \in L$.

Now suppose $f \in L$ and $k \in \mathbb{R}$. We will show that $kf \in L$. Let $v, w \in V$ and $\ell \in \mathbb{R}$. Then

$$(kf)(v+w) = k \cdot f(v+w) \text{ by def. of scalar multiplication in } W$$
$$= k[f(v)+f(w)] \text{ because } f \text{ is linear}$$
$$= k \cdot f(v) + k \cdot f(w) \text{ because } V \text{ is a vector space}$$
$$= (kf)(v) + (kf)(w) \text{ by def. of scalar multiplication in } W$$

Similarly, $(kf)(\ell v) = \ell \cdot (kf)(v)$.
We have shown that L is a subset of W; is nonempty; and is closed under addition and scalar multiplication. Therefore, L is a subspace of W.

(2) Let V be an n-dimensional vector space over $\mathbb{R}$, and let $B = \{\beta_1, \beta_2, \ldots, \beta_n\}$ be a basis of V.

(a) Prove: Every element in V can be expressed uniquely as a linear combination of B with coefficients in $\mathbb{R}$.
(b) Prove or disprove: $B' = \{\beta_1 + \beta_2, \beta_2, \beta_3, \ldots, \beta_n\}$ is also a basis for V.

Solution:
(a) Because B is a basis for V, B spans V. Hence every element in V can be expressed as a linear combination of B with coefficients in $\mathbb{R}$.
Now we show uniqueness. Suppose

$$c_1\beta_1 + c_2\beta_2 + \cdots + c_n\beta_n = a_1\beta_1 + a_2\beta_2 + \cdots + a_n\beta_n.$$

for some $c_1, c_2, \ldots, c_n, a_1, a_2, \ldots, a_n$. We will show that $c_1 = a_1, c_2 = a_2, \ldots, c_n = a_n$. Subtracting, we find that:

$$(c_1 - a_1)\beta_1 + (c_2 - a_2)\beta_2 + \cdots + (c_n - a_n)\beta_n = 0.$$

Because B is a basis, we have that B is linearly independent. Hence

$$c_1 - a_1 = c_2 - a_2 = \cdots = c_n - a_n = 0.$$

The result follows.
(b) Prove: It suffices to show that B' spans V and is linearly independent.
Claim (i): B' spans V: Let $\vec{v} \in V$. Since B is a basis,

$$\vec{v} = \sum_{i=1}^{n} a_i\beta_i,$$

where $a_i \in \mathbb{R}$ for all i. Hence,

$$\begin{aligned} \vec{v} &= a_1\beta_1 + a_2\beta_2 + a_3\beta_3 + \cdots a_n\beta_n \\ &= a_1(\beta_1 + \beta_2) + (a_2 - a_1)\beta_2 + a_3\beta_3 + \cdots + a_n\beta_n \end{aligned}$$

Therefore, B' spans V.
Claim (ii): B' is linearly independent.
Let

$$b_1(\beta_1 + \beta_2) + b_2\beta_2 + \cdots + b_n\beta_n = \vec{0}.$$

Then

$$b_1\beta_1 + (b_1 + b_2)\beta_2 + \cdots + b_n\beta_n = \vec{0}.$$

Since B is a basis, β_i are linearly independent, so $b_1 = b_1 + b_2 = b_3 = \cdots = b_n = 0$, implying $b_i = 0$ for all $1 \leq i \leq n$. Hence B' is linearly independent.

(3) For an $n \times n$ matrix A, the trace of A, denoted by $tr(A)$, is the sum of the n main diagonal entries of A. Suppose A is an $n \times n$ matrix with n eigenvalues (not necessarily distinct). Prove the following:

(a) The product of the n eigenvalues is equal to the determinant of A.

(b) The sum of the eigenvalues of A is equal to $tr(A)$.

Solutions: Let

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & \cdots & a_{2,n} \\ a_{3,1} & a_{3,2} & \cdots & \cdots & a_{3,n} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & \cdots & a_{n,n} \end{pmatrix}.$$

Then

$$tr(A) = \sum_{i=1}^{n} a_{i,i}.$$

The characteristic polynomial of A is:

$$det(A - \lambda I) = \begin{vmatrix} a_{1,1} - \lambda & a_{1,2} & \cdots & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} - \lambda & \cdots & \cdots & a_{2,n} \\ a_{3,1} & a_{3,2} & \cdots & \cdots & a_{3,n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n,1} & a_{n,2} & \cdots & \cdots & a_{n,n} - \lambda \end{vmatrix}. \qquad (*)$$

Let the eigenvalues of A be $\lambda_1, \lambda_2, \ldots \lambda_n$. Then

$$det(A - \lambda I) = (\lambda_1 - \lambda)(\lambda_2 - \lambda) \cdot \cdots \cdot (\lambda_n - \lambda). \qquad (**)$$

(a) Plug in $\lambda = 0$ to (**), we get

$$det(A) = \prod_{i=1}^{n} \lambda_i.$$

(b) The coefficient of $(-\lambda)^{n-1}$ in the right side of (**) is

$$\sum_{i=1}^{n} \lambda_i.$$

The co-factor expansion on the first row of the right-side of (*) shows that the coefficient of $(-\lambda)^{n-1}$ is the same as the coefficient of $(-\lambda)^{n-1}$ in

$$(a_{1,1} - \lambda)(a_{2,2} - \lambda)(a_{3,3} - \lambda) \cdots (a_{n,n} - \lambda),$$

which is

$$\sum_{i=1}^{n} a_{i,i}.$$

Therefore, $tr(A) = \sum_{i=1}^{n} \lambda_i$.

Groups

(1) Let a and b be elements of a finite group G. Show that $|ab| = |ba|$. (Here $|x|$ denotes the order of the element $x \in G$.)
Solution: First we prove that $|a^{-1}ba| = |b|$. Define $\phi : G \to G$ and $\psi : G \to G$ by

$$\phi(x) = a^{-1}xa \qquad \psi(x) = axa^{-1}$$

for all $x \in G$. Then ϕ and ψ are inverse automorphisms. Because automorphisms send subgroups to subgroups and generators to generators, $\langle b\rangle$ is isomorphic to $\phi(\langle b\rangle) = \langle\phi(b)\rangle = \langle a^{-1}ba\rangle$. In particular, $|b| = |\langle b\rangle| = |\langle a^{-1}ba\rangle| = |a^{-1}ba|$.
Now replace b by ab in $|b| = |a^{-1}ba|$ to get $|ab| = |a^{-1}(ab)a| = |ba|$.

(2) Let G be a finite group and $p \in \mathbb{N}$ a prime number. Show the following:

(a) If every element of G is contained in a subgroup of order p, then $|G| = p^n$ for some positive integer n.
(b) If every element of G is contained in a normal subgroup of order p, then G is abelian.

Solution:

(a) Let q be a prime number that divides $|G|$. Then, by Cauchy's Theorem, G contains an element of order q. Since elements in G have order 1 or p, we have $q = p$. So the only prime number that divides $|G|$ is p, and $|G|$ is a power of p.
(b) Let $h, k \in G$. If $h = 1$ or $k = 1$ then $hk = kh$ is immediate. Otherwise $H = \langle h\rangle$ and $K = \langle k\rangle$ are normal subgroups of order p. If $H = K$, then h and k commute because all groups of order p are abelian. If $H \neq K$, then $h^{-1}k^{-1}hk \in H \cap K = \{1\}$, so $h^{-1}k^{-1}hk = 1$, which can be written as $hk = kh$.

(3) Let $n \geq 2$ be an integer. Let A_n and S_n be the alternating and symmetric groups on n letters, respectively. Let $\tau \in S_n$ be a transposition. Define $\varphi\colon A_n \to S_n$ by $\varphi(\sigma) = \tau\sigma\tau$.

(a) Prove that φ is a homomorphism.
(b) Find the kernel of φ.
(c) Is φ injective? Prove that your answer is correct. Hint: Use your answer from (b).
(d) Is φ an isomorphism from A_n to S_n? Prove that your answer is correct. Hint: Consider the orders of these groups.

Solutions
(a) For all $\sigma_1, \sigma_2 \in A_n$, we have that $\varphi(\sigma_1\sigma_2) = \tau\sigma_1\tau\tau\sigma_2\tau = \tau\sigma_1\sigma_2\tau = \varphi(\sigma_1\sigma_2)$. Here we use that $\tau^2 = \iota$, because τ is a transposition. We use ι to denote the identity element of S_n.
(b) We have that

$$
\begin{aligned}
\ker(\varphi) &= \{\sigma \in A_n \mid \varphi(\sigma) = \iota\} \\
&= \{\sigma \in A_n \mid \tau\sigma\tau = \iota\} \\
&= \{\sigma \in A_n \mid \sigma\tau = \tau^{-1} = \tau\} \\
&= \{\sigma \in A_n \mid \sigma = \iota\} \\
&= \{\iota\}
\end{aligned}
$$

(c) By part (b), because the kernel is trivial, therefore φ is injective.
(d) The mapping φ cannot be bijective, because $|A_n| = n!/2 < n! = |S_n|$. So φ is not an isomorphism.

Synthesis: Linear algebra and group theory

(1)

(a) Let G be the group of all 2×2 matrices with entries from the real numbers. Here the group operation is addition of matrices. Define $f\colon G \to \mathbb{R}$ by $f(A) = \det A$. Is f a homomorphism? Prove that your answer is correct.

(b) Let H be the group of all 2×2 matrices with entries from the real numbers. Here the group operation is multiplication of matrices. Define $j\colon H \to \mathbb{R}$ by $j(A) = \det A$. Is j a homomorphism? Prove that your answer is correct.

Solution:
(a) No, it is not, because (for example):

$$
f\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = 1, \text{ but}
$$

$$
f\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right) + f\left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right) = 0 + 0 = 0 \neq 1.
$$

(b) Yes, j is a homomorphism, because $j(AB) = \det(AB) = \det A \det B = j(A)j(B)$ for all $A, B \in H$.

(2) Let m be a positive integer. Let $GL(m, \mathbb{R})$ be the set of all invertible $m \times m$ matrices with real entries. You may assume without proof that $GL(m, \mathbb{R})$ is a group under matrix multiplication. Let $A \in GL(m, \mathbb{R})$. Let $G = \{A^k \mid k \in \mathbb{Z}\}$ be the cyclic subgroup generated by A. You may assume without proof that there is a left group action of G on $\mathbb{R}^n$ given by matrix multiplication, namely the mapping $(B, w) \mapsto Bw$. Here we think of elements of $\mathbb{R}^n$ as column vectors. Suppose that v is an eigenvector of A. Let Y be the orbit of v under the action of the group G. Prove that the dimension of the span of Y is 1.

Solution: Because v is an eigenvector for A, we have that $v \neq 0$ and $Av = \lambda v$ for some $\lambda \in \mathbb{R}$. So $Y = \{A^k v \; k \in \mathbb{Z}\} = \{\lambda^k v \; k \in \mathbb{Z}\}$. Thus the span of Y is

$$
\{c_1\lambda^{k_1}v + \cdots + c_n\lambda^{k_n}v \mid c_1, \ldots, c_n \in \mathbb{R}, k_1, \ldots, k_n \in \mathbb{R}, n \in \mathbb{N}\} = \{cv \mid c \in \mathbb{R}\}.
$$

Because $v \neq 0$, it follows that $\{v\}$ is a basis for the span of Y. Hence the dimension of the span of Y is 1.

(3) Let $GL(n, \mathbb{R})$ be the group of $n \times n$ invertible matrices with entries in $\mathbb{R}$ under matrix multiplication.

- Let $SL(n, \mathbb{R})$ be the set of matrices in $GL(n, \mathbb{R})$ with determinant 1.
- Let $O(n, \mathbb{R})$ be the set of orthogonal matrices (that is, $A^t = A^{-1}$) in $GL(n, \mathbb{R})$.
- Let $SO(n, \mathbb{R}) = O(n, \mathbb{R}) \cap SL(n, \mathbb{R})$.

(a) Prove: $SL(n, \mathbb{R})$ is a normal subgroup of $GL(n, \mathbb{R})$.
(b) Prove: $GL(n, \mathbb{R})/SL(n, \mathbb{R})$ is isomorphic to $(\mathbb{R}^\star, \times)$, where $\mathbb{R}^\star = \mathbb{R} \setminus \{0\}$.
(c) Prove or disprove: $SO(n, \mathbb{R})$ is a normal subgroup of $O(n, \mathbb{R})$. (You may assume without proof that $O(n, \mathbb{R})$ is a subgroup of $GL(n, \mathbb{R})$.)

Solution:

(a) First, we show $SL(n, \mathbb{R})$ is a subgroup of $GL(n, \mathbb{R})$. Let $A, B \in SL(n, \mathbb{R})$. Then $det(A) = det(B) = 1$ and $det(A \times B) = det(A) \cdot det(B) = 1$. Hence, $A \times B \in SL(n, \mathbb{R})$. Moreover, $det(A^{-1}) = 1/det(A) = 1$. Hence $A^{-1} \in SL(n, \mathbb{R})$. Therefore, $SL(n, \mathbb{R})$ is a subgroup of $GL(n, \mathbb{R})$.

Second, we show $SL(n, \mathbb{R})$ is a normal subgroup of $GL(n, \mathbb{R})$. Let $A \in GL(n, \mathbb{R})$ and $B \in SL(n, \mathbb{R})$. Then $det(A \times B \times A^{-1}) = det(B) = 1$. Hence, $SL(n, \mathbb{R})$ is a normal subgroup of $GL(n, \mathbb{R})$.

Alternate solution: Do (b) first. Because $SL(n, \mathbb{R}) = ker(\phi)$, it follows that $SL(n, \mathbb{R})$ is a normal subgroup of $GL(n, \mathbb{R})$.

(b) Define a homomorphism from $GL(n, \mathbb{R})$ to non-zero reals $\mathbb{R}^\star$:

$$\phi : GL(n, \mathbb{R}) \to \mathbb{R}^\star,$$

by $\phi(A) = det(A)$. Since $det(A \times B) = det(A)det(B)$, it is clear the ϕ is a homomorphism. Moreover, ϕ is onto. Let r be any non-zero real. The $n \times n$ diagonal matrix with r in one entry in the diagonal and other entries 1 in the diagonal has determinant r. The kernel of ϕ is $SL(n, \mathbb{R})$. By the first isomorphism theorem, $GL(n, \mathbb{R})/ker(\phi) \cong \mathbb{R}^\star$, implying $GL(n, \mathbb{R})/SL(n, \mathbb{R}) \cong \mathbb{R}^\star$.

(c) Yes, it is true. Let $A \in SO(n, \mathbb{R})$ and $B \in O(n, \mathbb{R})$. Then $det(A) = 1$, $A^{-1} = A^t$, and $B^{-1} = B^t$. Putting together we have:

$$(BAB^{-1})^t = (B^{-1})^t A^t B^t = BA^{-1}B^{-1},$$

and

$$(BAB^{-1})^{-1} = B^t A^{-1} B^{-1}.$$

Hence, $(BAB^{-1})^t = (BAB^{-1})^{-1}$. Moreover, as $det(BAB^{-1}) = det(A) = 1$, we obtain $BAB^{-1} \in SO(n, \mathbb{R})$. Therefore, $SO(n, \mathbb{R})$ is a normal subgroup of $O(n, \mathbb{R})$.

Alternate solution: We know from (a) that $SL(n, \mathbb{R})$ is a normal subgroup of $GL(n, \mathbb{R})$. We're given that $O(n, \mathbb{R})$ is a subgroup of $GL(n, \mathbb{R})$. The result follows.