ALGEBRA COMPREHENSIVE EXAM

FALL 2004

Answer 5 questions only. You must answer at least one from each of Groups, Rings, and Fields. Please show work to support your answers.

GROUPS

1. Let H and N be subgroups of a finite group G, N normal in G. Suppose that $[G : N]$ is finite and $|H|$ is finite, and $([G : N], |H|) = 1$. Prove that HN.

2. Assume $|G| = p^3$ (p a prime).

(a) Show $|Z(G)| > 1$.

(b) Prove that if H is non-abelian, then $|Z(G)| = p$.

3. Let P be a Sylow p-subgroup of G. Assume that $P \lhd N \lhd G$. Show that $P \lhd G$.

RINGS

1. Let R be a commutative ring with identity. Assume $1 = e + f$, and $ef = 0$. Define $\Phi : R \to R$ by $\Phi(x) = ex$. Prove:

(a) e is an idempotent (i.e. $e^2 = e$).

(b) Φ is a ring homomorphism.

(c) e is the identity of $\Phi(R)$ (the image of Φ).

2. Let R be a nonzero ring such that $x^2 = x$ for all $x \in R$. Show that R is commutative and has characteristic 2.

3. Prove that if F is a field then every ideal of the ring $F[x]$ is principal.

FIELDS

1. Show that the group of automorphisms of the rational numbers Q is trivial.

2. Let E be the splitting field of $x^6 - 3$ over the rationals Q.

(a) Find $[E : Q]$. Explain.

(b) Show that the Galois group $\mathcal{G}(E/Q)$ is not abelian.

3. (a) Let F be a field and let $f(x) \in F[x]$ with $\deg(f(x)) = n > 0$. Prove that $f(x)$ has at most n roots in F.

(b) Let F be a field and let $f(x)$ and $g(x)$ be elements of $F[x]$ with $\deg(f(x))$ and $\deg(g(x))$ each at most n. Suppose there exist $a_1, a_2, a_3, \ldots, a_{n+1} \in F$ such that $f(a_i) = g(a_i)$ for $1 \leq i \leq n + 1$. Prove that $f(x) = g(x)$.