ALGEBRA COMPREHENSIVE EXAMINATION
SPRING 2001

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Answer 5 questions only. You must answer *at least one* from each of Groups, Rings, and Fields.
Please show work to support your answers.

GROUPS

1. Let p be a prime and G be a finite p-group with center Z(G).
   (a) Show that $Z(G) \neq \{e\}$
   (b) If N is a normal subgroup with $|N| = p$, prove that $N \subseteq Z(G)$.

2. Prove that any group of order 255 is cyclic.

3. Let G be an group of order 405 ($= 3^4 \cdot 5$). Prove that G is solvable.

RINGS

1. Let R be a commutative ring with identity and let I be an ideal of R. Define
   $\alpha(I) = \{x \in R \mid \exists\, n \geq 1, \text{ with } x^n \in I\}$ and prove that:
   (a) $\alpha(I) \supseteq I$,
   (b) $\alpha(I)$ is an ideal of R, and
   (c) $\alpha(\alpha(I)) = \alpha(I)$.

2. Let R be a ring with identity and assume that $x \in R$ has a right inverse. Prove that the following are equivalent:
   (a) x has more than one right inverse,
   (b) x ix not a unit, and
   (c) x is a left zero-divisor.

3. Let $M = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| a,b,c,d \in R \right\}$

Where R is the set of real numbers with the usual matrix operations.
   (a) Prove that M is not a field.
   (b) Prove that an element A of M is a zero-divisor $\Leftrightarrow$ $\det A \neq 0$

FIELDS

1. Let E be the splitting field of $x^6 - 3$ over the rationals $\mathbf{Q}$.
   (a) Find $[E : \mathbf{Q}]$, and explain.
   (b) Show that the Galois group $\mathrm{Gal}(E/\mathbf{Q})$ is not abelian.

2. Prove that "algebraicness" is transitive; i.e., if E, F, and K is a tower of fields with F algebraic over E and K algebraic over F, then K is algebraic over E.

3. Let $E = \mathbf{Q}(\sqrt{3}, \sqrt{5})$ and $\alpha .= \sqrt{3} + \sqrt{5}$
   Prove: (a) $[E : \mathbf{Q}] = 4$.
   (b) $E = \mathbf{Q}(\alpha)$
   (c) Describe the Galois group $G(E/\mathbf{Q})$.