ALGEBRA COMPREHENSIVE EXAMINATION

Spring 2002

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Answer 5 questions only. You must answer *at least one* from each of Groups, Rings, and Fields.
Be sure to show enough work that your answers are adequately supported.

GROUPS

1. Let p and q be prime natural numbers with $p < q$
   Prove: (a) A group of order p^2 must be abelian.
   (b) A group of order p^2q must be solvable.

2. Characterize all groups of order 8 and justify your answer.

3. Let G be a group of order 231. Show that the Sylow 11-subgroup is a subgroup of the center of G.

RINGS

1. Let R and S be rings and let ϕ:R→S be a ring homomorphism. Prove:
   (a) If R has an identity, then ϕ[S] = Im(ϕ) has an identity.
   (b) If R is commutative, then ϕ[S] = Im(ϕ) is commutative.
   (c) If ϕ is 1-1 and S is a field, then R is an integral domain.
   (d) If R is an integral domain and ϕ is onto, decide whether or not S must be an integral domain and prove your result.

2. Let R be a commutative ring with identity and I an ideal of R.

   DEF: $\sqrt{I} = \{x{:}\, x^n \in I, some\ \text{n}\}$ Prove:

   a.. $\sqrt{I}$ is an ideal of R.

   b. If $I \subseteq J$ are ideals then $\sqrt{I} \subset \sqrt{J}$

   c. $\sqrt{\sqrt{I}} = \sqrt{I}$

   d. If I and J are ideals of R then $\sqrt{I \cap J} = \sqrt{I} \cap \sqrt{J}$

3. Let R = $\{a/b \in \mathbf{Q} \mid a, b \in \mathbf{Z}$ and $2 \nmid b\}$ with the usual rational number operations.
   (a) Prove that R is an integral domain.
   (b) Find U(R), the group of units (invertible elements) of R.
   (c) Prove that R\U(R) [= R - U(R)] is the unique maximal ideal in R.

FIELDS

1. Let F be a field and K an extension field of F.
   Define $\mathcal{G}$ (K/F), the Galois group of K over F.
   Describe explicitly the elements of $\mathcal{G}$ ($\mathbf{Q}(\sqrt{5}+\sqrt{2})/\mathbf{Q}$) where $\mathbf{Q}$ is the field of rationals.
   Identify $\mathcal{G}$ ($\mathbf{Q}(\sqrt{5}+\sqrt{2})/\mathbf{Q}$). up to isomorphism. That is, find a well-known group which is isomorphic to $\mathcal{G}$.

2. Let K, L, F be fields with K ⊇ L ⊇ F , [L:F] = m, [K:L] = n. Prove that [K:F] = mn.

3. Let $\mathbf{Q}$ be the field of rationals and let $p(x) = x^3 + 2x^2 + 6$. Prove that $p(x)$ is irreducible and, if α is a root of $p(x)$, express $1/(3\alpha - 2)$ as a linear combination of the set $\{1, \alpha, \alpha^2\}$.