ALGEBRA COMPREHENSIVE EXAMINATION
Spring 2003
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Answer five questions only. You must answer *at least one* from each of Groups, Rings and Fields. Please show work to support your answers.

GROUPS

1. Let A, B and C be normal subgroups of a group G with $A \subseteq B$. If $A \cap C = B \cap C$ and $AC = BC$ then prove that $A = B$.
2. Let G be a finite group with identity e, and such that for some fixed integer $n > 1, (xy)^n = x^n y^n$ for all $x, y \in G$. Let $G_n = \{z \in G : z^n = e\}$ and $G^n = \{x^n : x \in G\}$. Prove that both G_n, and G^n are normal subgroups of G and that $|G^n| = [G : G_n]$.
3. Prove:
   a. A group of order 45 is abelian.
   b. A group of order 275 is solvable.

RINGS

1. Let R be a commutative ring with unity and let I be an ideal of R. Define
   $$\sqrt{I} = \{x \in R : \exists\, n \geq 1 \text{ such that } x^n \in I\}.$$
   Prove that
   (a) $\sqrt{I} \supseteq I$,
   (b) $\sqrt{I}$ is an ideal of R,
   (c) $\sqrt{\sqrt{I}} = \sqrt{I}$, and
   (d) $\sqrt{A \cap B} = \sqrt{A} \cap \sqrt{B}$ where A and B are ideals of R.
2. Let R be a commutative ring with identity 1 and let M be an ideal of R. Prove that M is a maximal ideal $\iff \forall\, r \in R - M,\ \exists\, x \in R$ such that $1 - rx \in M$.
3. Let D be an Euclidean domain. Let a, b nonzero elements of D and d their GCD. Prove that $d = ax + by$ for some $x, y \in D$.

FIELDS

1. For some prime p, let $f(x)$ be an irreducible polynomial in $Z_p[x]$, the ring of polynomials with coefficients in Z_p. Prove that $f(x)$ divides $x^{p^n} - x$ for some n.
2. Let Q be the field of rational numbers and let E be the splitting field of $x^4 - 2$ over Q.
   (a) Find $[E : Q]$ and explain your answer.
   (b) Show that the Galois group Gal (E/Q) is not abelian.
3. Let F be the Galois field with 2^n elements. Prove that any $\alpha \in F$ has a square root in F; that is, $x^2 = \alpha$ is solvable in F.