ALGEBRA COMPREHENSIVE EXAMINATION

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Answer 5 questions only. You must answer *at least one* from each of groups, rings, and fields. Be sure to show enough work that your answers are adequately supported.

GROUPS

1. Let G be an abelian group, $H = \{a^2 \mid a \in G\}$ and $K = \{a \in G \mid a^2 = e\}$. Prove that $H \cong G/K$.

2. Assume $G = HZ(G)$, where H is a subgroup of G and $Z(G)$ is the center of G. Show:
   a. $Z(H) = H \cap Z(G)$
   b. $G' = H'$ (Where G' is the derived group of G)
   c. $G/Z(G) \cong H/Z(H)$

3. Prove:
   a. A group of order 80 need not be abelian (twice) by exhibiting two non-isomorphic non-abelian groups of order 80 (with verification).
   b. A group of order 80 must be solvable.

RINGS

1. Let R be a ring with ideals A and B.
   a. Define a natural function φ: $R/A \cap B \to R/A \times R/B$ and show that it is a ring homomorphism.
   b. Calculate Ker(φ), the kernel of φ.
   c. Prove that if $R = A + B$, then φ is an isomorphism.
   d. Show that the converse of (c) is false.

2. Let R be a subring of a field F such that, for every $x \in F$, either $x \in R$ or $x^{-1} \in R$. Prove that the ideals of R are linearly ordered; i.e., if I and J are ideals of R, then either $I \subseteq J$ or $J \subseteq I$.

3. Let $M_2(\mathbf{Q})$ be the ring of 2×2 all matrices with rational enties. Prove:
   a. $M_2(\mathbf{Q})$ has no nontrivial ideals.
   b. $M_2(\mathbf{Q})$ has an identity but is not a field.

FIELDS

1. Find the minimal polynomial for $\alpha = \sqrt{5 + \sqrt{2}}$ over the field of rationals **Q** and prove it is minimal.

2. Let GF(p^n) denote the Galois field with p^n elements.
   (a) Prove that GF(p^a) $\subseteq$ GF(p^b) iff a divides b.
   (b) Prove that GF(p^a) $\cap$ GF(p^b) = GF(p^d), where d = gcd(a, b).

3. Let F be a finite field of $n = p^m$ elements. Find necessary and sufficient conditions to insure that $f(x) = x^2 + 1$ has a root in F; i.e., f is not irreducible over F.