Algebra Comprehensive Exam Spring 2007

Brookfield, Krebs*, Shaheen

Answer five (5) questions only. You must answer *at least one* from each of groups, rings, and fields. Be sure to show enough work that your answers are adequately supported.

Groups

(1) Show that any group of order 441 has a normal subgroup of order 49.

(2) Let $\phi : G \to H$ be a group homomorphism, where G and H are finite groups such that the order of G and the order of H are relatively prime. Show that ϕ is trivial. (That is, show that $\phi(g) = e_H$ for all $g \in G$, where e_H is the identity element of H.)

(3) Suppose that G is a group of order p^n, where p is a prime number and n is a positive integer. Prove that if the center of G has order p, then G contains no more than $p^{n-1}+p-1$ conjugacy classes.

Rings

(1) Let R be a ring with identity 1 and $a, b \in R$ such that $ab = 1$. Let $X = \{x \in R \mid ax = 1\}$. Show the following:

a. If $x \in X$, then $b + 1 - xa \in X$.

b. If $\phi : X \to X$ is defined by $\phi(x) = b + 1 - xa$ for $x \in X$, then ϕ is injective (one-to-one).

c. X contains either exactly one element or infinitely many elements. Hint: Recall the Pigeonhole Principle—an injective (one-to-one) function from a finite set to itself is surjective (onto).

(2) Let F be a field and let $p \in F[x]$ such that $p \neq 0$. Show that the ideal (p) is maximal in $F[x]$ iff p is irreducible over F.

(3) Let $\mathbb{Z}[x]$ be the ring of polynomials in x with integer coefficients. Prove that $\mathbb{Z}[x]$ is not a Euclidean domain. (Hint: Consider the ideal I of all polynomials in $\mathbb{Z}[x]$ whose constant terms are even.)

Fields

(1) Let $\mathbb{F}_q$ be a finite field with q elements, where $q = p^r$ for some odd prime number p and some positive integer r. Show that $a \in \mathbb{F}_q^*$ has a square root in $\mathbb{F}_q^*$ (that is, $x^2 = a$ has a solution in $\mathbb{F}_q^*$) iff $a^{\frac{q-1}{2}} = 1$. (Here $\mathbb{F}_q^*$ denotes the multiplicative group of nonzero elements in $\mathbb{F}_q$.)

(2) Let $\sigma = e^{2\pi i/7} \in \mathbb{C}$, and let $F = \mathbb{Q}(\sigma)$. Describe the Galois group of F over $\mathbb{Q}$. Explain what theorems you are using. (Here $\mathbb{C}$ denotes the field of complex numbers, and $\mathbb{Q}$ denotes the field of rational numbers.)

(3) Let K be an extension field of F and $\alpha \in K$. Show that if $F(\alpha) = F(\alpha^2)$, then α is algebraic over F.