California State University – Los Angeles

Department of Mathematics

Master's Degree Comprehensive Examination

Analysis   Spring 2021

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Do at least five of the following seven problems. All problems count equally. If you attempt more than five, the best five will be used.

(1) Write in a fairly soft pencil (number 2) (or in ink if you wish) so that your work will duplicate well. There should be a supply available.
(2) Write on one side of the paper only.
(3) Begin each problem on a new page.
(4) Assemble the problems you hand in in numerical order.

Exams are graded anonymously, so put your name only where directed and follow any instructions concerning identification code numbers.

SECTION 1 – Do two (2) problems from this section. If you attempt all three, then the best two will be used for your grade.

Spring 2021 #1. Compute the $\inf, \sup, \max$ and $\min$ (when they exist) of the following sets:

(a) $\{x \in \mathbb{Q} |\ x^5 < 243\}$.

(b) $\{x \in \mathbb{R} \setminus \mathbb{Q} |\ x^5 < 243\}$.

(c) $\{1 + \frac{(-1)^n}{n} |\ n \in \mathbb{N}\}$.

Spring 2021 #2. Using the $\varepsilon - \delta$ definition show that

$$\lim_{x\to 0} \frac{1}{(x-1)^2} = 1.$$

Is the function $\dfrac{1}{(x-1)^2}$ continuous at zero? Justify your answer.

Spring 2021 #3. Consider the set $S = [0, \infty)$. Consider the open cover:

$$X = \{(n-2, n) |\ n \in \mathbb{N}\}.$$

(a) Is X a cover of S? Justify your answer.

(b) Prove that X contains no finite subcover of S.

(c) Is S closed? Is it bounded? Justify your answer.

SECTION 2 – Do three (3) problems from this section. If you attempt more than three, then the best three will be used for your grade.

Spring 2021 #4. We define a transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ by

$$Tx = Cx + b,$$

where $C = (c_{ij})$ is a real $n \times n$ matrix and $b \in \mathbb{R}^n$ is given.

(a) If we equip $\mathbb{R}^n$ with the metric $d(x, y) = \max_{1\le i\le n} |x_i - y_i|$, under what general condition on the matrix C will T be a proper contraction? Justify your answer.

(b) Repeat Part (a) with the metric $d(x, y) = \left(\sum_{i=1}^{n} |x_i - y_i|^2\right)^{1/2}$.

(c) With either of the metric from (a) or (b), describe an iteration process for solving the linear system of equations

$$x = Cx + b$$

by specifying the transformation to be iterated and explaining how this leads to a solution.

Spring 2021 #5. Let $p_0(x) = 1$ and $p_1(x) = x$, and let $\mathcal{P}_1$ be the subspace of the space $C([0,2])$ of all continuous functions on $[0,2]$ spanned by p_0 and p_1.

(a) Find a basis for $\mathcal{P}_1$ which is orthonormal with respect to the inner product $\langle f, g\rangle = \int_0^2 f(t)\overline{g(t)}\, dt$.

(b) Use the results of Part (a) to find the function $f(x) = ax + b$ in $\mathcal{P}_1$ which makes the quantity

$$J(f) = \int_0^2 |x^2 - f(x)|^2\, dx$$

as small as possible.

Spring 2021 #6. If d is a metric on a vector space $X \neq \{0\}$ which is obtained from a norm $\|\cdot\|$, and $\tilde{d}$ is defined by

$$\tilde{d}(x,x) = 0, \qquad \tilde{d}(x,y) = d(x,y) + 1 \quad (x \neq y).$$

(a) Show that $\tilde{d}$ is a metric on X.

(b) Show that $\tilde{d}$ cannot be obtained from a norm.

Spring 2021 #7. Let $f(t) = t^2$ for $t \in [-\pi, \pi]$, and extend it to be 2π-periodic on $\mathbb{R}$.

(a) Find the Fourier series for $f(t)$ in the trigonometric form.

(b) Use the result of Part (a) to show that

$$1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \cdots = \frac{\pi^4}{90}.$$