California State University – Los Angeles

Department of Mathematics

Master's Degree Comprehensive Examination

Analysis   Fall 2021

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Do at least five of the following seven problems. All problems count equally. If you attempt more than five, the best five will be used.

(1) Write in a fairly soft pencil (number 2) (or in ink if you wish) so that your work will duplicate well.
(2) Write on one side of the paper only.
(3) Begin each problem on a new page.
(4) Assemble the problems you hand in in numerical order.

Exams are graded anonymously, so put your name only where directed and follow any instructions concerning identification code numbers.

SECTION 1 – Do two (2) problems from this section. If you attempt all three, then the best two will be used for your grade.

Fall 2021 #1. Prove the following statements using the $\varepsilon - N$ definition.

(a) $x_n \to x$, $y_n \to x \Longrightarrow x_n + y_n \to 2x$.

(b) $x_n \to x$, $x_n \to y \Longrightarrow x = y$.

(c) Let $\varepsilon = 0.001$, find N such that it shows that the sequence $x_n = \frac{n}{5n+1}$ converges to $\frac{1}{5}$.

Fall 2021 #2. For each of the following statement, determine whether it is true or false. If it is true, prove it. If it is false, provide a counter example.

(a) Every convergent sequence is bounded.

(b) If $A \subset B$, then $\sup A \leq \sup B$.

(c) If $\sup A = \sup B$ and $\inf A = \inf B$, then $A = B$.

Fall 2021 #3. Prove the following limit using the $\varepsilon - \delta$ definition.

$$\lim_{x \to 1} \frac{4}{(x+1)^2} = 1.$$

Determine whether or not the function $\dfrac{4}{(x+1)^2}$ is continuous at $x = 1$. Justify your answer.

SECTION 2 – Do three (3) problems from this section. If you attempt more than three, then the best three will be used for your grade.

Fall 2021 #4. Prove or find a counter example to the following statements.

(a) A normed space is necessarily a metric space.

(b) A metric space is necessarily a normed space.

(c) An inner product space is necessarily a normed space.

(d) A normed space is necessarily an inner product space.

Fall 2021 #5. Let $\mathcal{X} = C([0,1])$ be the space of continuous complex valued functions on $[0,1]$. Let a be a real constant with $0 < a < 1$.

Define $\phi : \mathcal{X} \to \mathbb{C}$ by:

$$\phi(f) = \int_0^a (x^2+1) f(x)\, dx$$

(a) Show $\langle f, g\rangle = \int_0^1 f(x)\overline{g(x)}dx$ is an inner product on $\mathcal{X}$.

(b) Show that ϕ is linear.

(c) Show that ϕ is continuous with respect to the norm associated to the inner product of Part (a).

Fall 2021 #6.

(a) Show that $\{\cos(2x), \sin(2x)\}$ is an orthonormal family with respect to the inner product

$$\langle f, g\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\overline{g(t)}\, dt.$$

(b) Use Part (a) to find the function $f(x) = a\cos(2x) + b\sin(2x)$ which makes the quantity

$$J(f) = \int_{-\pi}^{\pi} |1 + 2x - f(x)|^2\, dx$$

as small as possible.

Fall 2021 #7. For f in the space $C([0,1])$ of continuous functions on the interval $[0,1]$, define Tf by

$$(Tf)(x) = \sin(x) + \lambda \int_0^x (x - t^2) f(t)\, dt$$

(a) Find a range of values of λ for which T is a proper contraction with respect to the L^2-norm on $C([0,1])$. Justify your answer.

(b) Describe the iterative process for solving the integral equation $f(x) = (Tf)(x)$ by specifying the transformation to be iterated and explaining how this leads to a solution.

(c) Show that solutions f to the equation $f(x) = (Tf)(x)$ satisfy the ordinary differential equation

$$f''(x) + \lambda x(x-1)f'(x) + 2\lambda(x-1)f(x) = -\sin x.$$