**California State University – Los Angeles**

**Department of Mathematics**

**Master's Degree Comprehensive Examination**

**Analysis Spring 2023**

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Do at least two (2) problems from Section 1 below, and at least three (3) problems from Section 2 below. All problems count equally. If you attempt more than two problems from Section 1, the best two will be used. If you attempt more than three problems from Section 2, the best three will be used.

(1) Write in a fairly soft pencil (number 2) (or in ink if you wish) so that your work will duplicate well. There should be a supply available.
(2) Write on one side of the paper only.
(3) Begin each problem on a new page.
(4) Assemble the problems you hand in in numerical order.

**Exams are graded anonymously, so put your name only where directed and follow any instructions concerning identification code numbers.**

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**SECTION 1 – Do two (2) problems from this section. If you attempt all three, then the best two will be used for your grade.**

**Spring 2023 #1.** Use the definition of continuity to show that the function

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{otherwise} \end{cases}$$

is not continuous anywhere on $\mathbb{R}$.

*Proof.* See `https://proofwiki.org/wiki/Dirichlet_Function_is_Discontinuous`. $\square$

**Spring 2023 #2.** Let $a_n$ be the sequence

$$a_n = \begin{cases} 1 - \frac{1}{2^n} & \text{if } n \text{ is odd,} \\ 0 & \text{otherwise.} \end{cases}$$

Prove that

$$\liminf a_n = 0 \quad \text{and} \quad \limsup a_n = 1.$$

*Proof.* For all positive integers $k$, let

$$S_k = \{a_k, a_{k+1}, a_{k+2}, \dots\} = \{a_j \mid j \geq k\}.$$

So:

$$S_1 = \{a_1, a_2, a_3, \dots\} = \left\{1 - \frac{1}{2}, 0, 1 - \frac{1}{8}, 0, 1 - \frac{1}{32}, \dots\right\}$$

$$S_2 = \{a_2, a_3, a_4, \dots\} = \left\{0, 1 - \frac{1}{8}, 0, 1 - \frac{1}{32}, 0, \dots\right\}$$

$$S_3 = \{a_3, a_4, a_5, \dots\} = \left\{1 - \frac{1}{8}, 0, 1 - \frac{1}{32}, 0, 1 - \frac{1}{2^7}, \dots\right\}$$

Etc.

Note that for all $k$, we have that $\sup S_k = 1$ and $\inf S_k = 0$.

Therefore

$$\liminf a_n = \sup\{\inf S_k \mid k \in \mathbb{N}\} = \sup\{0\} = 0, \text{ and}$$

$$\limsup a_n = \inf\{\sup S_k \mid k \in \mathbb{N}\} = \sup\{1\} = 1. \qquad \square$$

**Spring 2023 #3.** Let $(a_n)$ be a sequence of real numbers.

a. State the definition of a Cauchy sequence.
b. Prove that if the sequence $(a_n)$ converges, then it is a Cauchy sequence.

Solution:

a. A sequence $a_n$ is Cauchy if, for every $\epsilon > 0$, there exists a natural number $N$ such that

$$|a_n - a_m| < \epsilon$$

whenever $n, m \geq N$.

b. Assume that $a_n$ is a convergent sequence. Then there exists a number $L$ such that, for any $\epsilon > 0$, there exists a natural number $N$ for which

$$|a_n - L| < \frac{\epsilon}{2}$$

whenever $n \geq N$. Consider now the quantity

$$|a_n - a_m|$$

for positive integers $m, n$. Applying the Triangle Inequality, we have that

$$\begin{aligned} |a_n - a_m| &= |(a_n - L) + (L - a_m)| \\ &\leq |a_n - L| + |a_m - L|. \end{aligned}$$

Since we assumed the sequence was convergent, then for any $\epsilon > 0$, we can find $N$ so that

$$|a_n - L| < \frac{\epsilon}{2} \quad \text{and} \quad |a_m - L| < \frac{\epsilon}{2}$$

for $n, m \geq N$, which implies that

$$|a_n - a_m| < \epsilon$$

for $n, m \geq N$. It follows that $a_n$ is a Cauchy sequence.

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**SECTION 2 – Do three (3) problems from this section. If you attempt more than three, then the best three will be used for your grade.**

**Spring 2023 #4.** Let $\mathcal{H}$ be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$, and let $y, z \in \mathcal{H}$. Define $T \colon \mathcal{H} \to \mathcal{H}$ by

$$T(x) = \langle x, y \rangle z.$$

a. Prove that $T$ is a linear transformation.
b. Prove that $T$ is bounded.
c. Prove that $T$ is continuous. Hint: Use parts (a) and (b).
d. Prove that $\|T\| \leq \|y\|\|z\|$. Here $\|T\|$ denotes the operator norm of $T$. Hint: Use your answer to part (b).

Solutions:

a. For all $x_1, x_2 \in \mathcal{H}$, we have that

$$T(x_1 + x_2) = \langle x_1 + x_2, y\rangle z = \langle x_1, y\rangle z + \langle x_2, y\rangle z = T(x_1) + T(x_2).$$

Here we used linearity of inner products in the first factor, as well as a distributive property (which holds in all vector spaces, including Hilbert spaces).

Let $\mathbb{F}$ be the field over which $\mathcal{H}$ is a vector space.

For all $x \in \mathcal{H}, \lambda \in \mathbb{F}$, we have that

$$T(\lambda x) = \langle \lambda x, y\rangle z = \lambda\langle x, y\rangle z = \lambda T(x).$$

Here we used linearity of inner products in the first factor, as well as the associative property for scalar multiplication in vector spaces.

b. Let $v \in \mathcal{H}$ such that $\|v\| = 1$. Then

$$\begin{aligned}
\|T(v)\| &= \|\langle v, y\rangle z\| \\
&= |\langle v, y\rangle| \cdot \|z\| && \text{by def. of norm} \\
&\leq \|v\| \cdot \|y\| \cdot \|z\| && \text{by the Cauchy-Schwarz inequality} \\
&= \|y\| \cdot \|z\|
\end{aligned}$$

c. Every bounded linear transformation is continuous.

d. The answer to (b) shows that this is true.

**Spring 2023 #5.** Let $f(x) : \mathbb{R} \to \mathbb{R}$ be the function defined by setting

$$f(x) = \begin{cases} 1 + \frac{x}{\pi} & \text{if } -\pi \leq x < 0, \\ 1 - \frac{x}{\pi} & \text{if } 0 \leq x < \pi, \end{cases}$$

and then extending the result $2\pi$-periodically.

a. Find the trigonometric Fourier series for $f(x)$.

b. Use the result from part (a) to find the sum of the series

$$\sum_{n=1}^{\infty} \frac{1-(-1)^n}{n^2},$$

and prove that your answer is correct.

Solution:

a. In general, the Fourier series for a function $f(x)$ is given by

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx)\right),$$

where the $a_n$ and $b_n$ are given by

$$\begin{aligned} a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\ dx \\ a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\ dx \\ b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\ dx \end{aligned}$$

for $n = 1, 2, 3, \ldots$ (Note: Different books use slightly different formulas for this.) Since $f(x)$ is an even function, the product $f(x)\sin(nx)$ is odd, which forces $b_n = 0$ for all $n$. As for the $a_n$, you should be able to check that $a_0 = 1$ and

$$a_n = \frac{2\left(1-(-1)^n\right)}{\pi^2 n^2}$$

for $n = 1, 2, 3, \ldots$ We thus conclude that the trigonometric Fourier series for $f(x)$ is

$$\frac{1}{2} + \sum_{n=1}^{\infty} \left(\frac{2\left(1-(-1)^n\right)}{\pi^2 n^2}\right) \cos(nx).$$

b. Observe that $f$ is continuous. (You can check at $x = 0$ that the two "pieces" in the graph of $f$ match up.)

Moreover, putting aside the single term $a_0/2 = 1/2$, the sum of the other Fourier coefficients is

$$\frac{4}{\pi^2} + \frac{4}{\pi^2 3^2} + \frac{4}{\pi^2 5^2} + \cdots = \frac{4}{\pi^2}\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right)$$

The factor inside the parentheses includes every other term of the $p$-series $1/n^2$, which is convergent. So by the Direct Comparision Test for series, we have that the sum of the absolute values of the Fourier coefficients converges. Therefore the Fourier series for $f$ converges absolutely to $f$. Therefore it converges pointwise to $f$ as well.

Thus, using the result from part (a) with $x = 0$, we obtain

$$1 = f(0) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2\left(1 - (-1)^n\right)}{\pi^2 n^2},$$

hence

$$\frac{\pi^2}{4} = \sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n^2}.$$

**Spring 2023 #6.** Let $H$ be a Hilbert space. Suppose that $H$ is the orthogonal direct sum of two closed subspaces $M$ and $N$. Moreover, suppose that $E$ is an orthonormal basis for $M$, and suppose that $F$ is an orthonormal basis for $N$. Prove that $E \cup F$ is an orthonormal basis for $H$.

Solution:

See #3 from Quiz 5 Take 1 here:

`https://drive.google.com/file/d/1nJOpjo3RLl-LHZquBTLlemthhKPYzarY/view?pli=1`

**Spring 2023 #7.** Let $I$ be the interval $[a, b]$ for some $a < b$, and let

$$C(I) = \{f : I \to \mathbb{R} : f \text{ is continuous}\}.$$

For $f, g \in C(I)$, define

$$d(f, g) = \sup_{x \in I} |f(x) - g(x)|.$$

Show that $d$ defines a metric on $C(I)$.

Solution: To show that $d$ is a metric, we must show that for any functions $f, g, h \in C(I)$, the following properties hold:

a. $d(f, g) \geq 0$;
b. $d(f, g) = 0$ if and only if $f = g$;
c. $d(f, g) = d(g, f)$;
d. $d(f, g) \leq d(f, h) + d(h, g)$.

In addition, we must show that $d$ is well-defined function from $C(I) \times C(I)$ to $\mathbb{R}$, i.e., that for all continuous functions $f, g$ on $I$, we have that $d(f, g)$ is a (finite) real number. (Side comment: If $I$ were an open instead of closed interval, this property would fail. Consider functions like $1/x$.)

Because $f$ and $g$ are continuous, so is the function $f - g$ defined by $(f-g)(x) = f(x)-g(x)$. The absolute value of a continuous function is continuous, so the function $|f(x) - g(x)|$ is also continuous. Moreover, $I$ is compact, because $I$ is a closed interval. Thus, by the Extreme Value Theorem, we have that $|f(x) - g(x)|$ attains a maximum on $I$. Hence $d(f, g)$ is a (finite) real number.

Now we prove properties (a), (b), (c), and (d).

Let $f, g \in C(I)$. It is obvious that

$$0 \leq |f(x) - g(x)| \leq \sup_{x \in I} |f(x) - g(x)|$$

by definition of supremum. Thus, property (a) holds.

Next, assume that $d(f, g) = 0$, so that

$$0 = \sup_{x \in I} |f(x) - g(x)|.$$

Thus, for any $x \in I$, we have

$$|f(x) - g(x)| = 0,$$

which implies that $f = g$. Conversely, if we assume that $f = g$, then

$$d(f, g) = \sup_{x \in I} |f(x) - g(x)| = \sup_{x \in I} |f(x) - f(x)| = 0.$$

Thus, property (b) holds.

To show that property (c) holds, we observe that

$$|f(x) - g(x)| = |g(x) - f(x)|$$

by properties of absolute value. Thus, we may conclude that

$$d(f, g) = \sup_{x \in I} |f(x) - g(x)| = \sup_{x \in I} |g(x) - f(x)| = d(g, f).$$

Finally, we show that property (d) holds. First, we observe that by the Triangle Inequality, we have

$$\begin{aligned} |f(x) - g(x)| &\leq |f(x) - h(x)| + |h(x) - g(x)| \\ &\leq \sup_{x \in I} |f(x) - h(x)| + \sup_{x \in I} |h(x) - g(x)| \\ &= d(f, h) + d(h, g) \end{aligned}$$

by definition of the supremum. Thus, $d(f, h) + d(h, g)$ is an upper bound for $|f(x) - g(x)|$. By definition, it must be greater than the least upper bound, from which we obtain that

$$d(f, g) = \sup_{x \in I} |f(x) - g(x)| \leq d(f, h) + d(h, g),$$

as desired.