# Antipodal Labelings for Cycles

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**Abstract**

Let $G$ be a graph with diameter $d$. An antipodal labeling of $G$ is a function $f$ that assigns to each vertex a non-negative integer (label) such that for any two vertices $u$ and $v$, $|f(u) - f(v)| \geq d - d(u,v)$, where $d(u,v)$ is the distance between $u$ and $v$. The span of an antipodal labeling $f$ is $\max\{f(u) - f(v) : u, v \in V(G)\}$. The antipodal number for $G$, denoted by $\mathrm{an}(G)$, is the minimum span of an antipodal labeling for $G$. Let $C_n$ denote the cycle on $n$ vertices. Chartrand et al. [4] determined the value of $\mathrm{an}(C_n)$ for $n \equiv 2 \pmod 4$. In this article we obtain the value of $\mathrm{an}(C_n)$ for $n \equiv 1 \pmod 4$, confirming a conjecture in [4]. Moreover, we settle the case $n \equiv 3 \pmod 4$, and improve the known lower bound and give an upper bound for the case $n \equiv 0 \pmod 4$.

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# 1 Introduction

Radio $k$-labeling was motivated by the frequency assignment problem (cf. [7]). Let $k$ be a positive integer. A *radio $k$-labeling* (or *$k$-labeling* for short) for a graph $G$ is a function, $f : V(G) \to \{0, 1, 2, \cdots\}$, such that the following is satisfied for any vertices $u$ and $v$:

$$|f(u) - f(v)| \geq k + 1 - d(u, v).$$

where $d(u, v)$ denotes the distance between $u$ and $v$. The *span* of such a function $f$, denoted by $\mathrm{sp}(f)$, is defined as $\mathrm{sp}(f) = \max\{f(u) - f(v) : u, v \in V(G)\}$. The minimum span over all $k$-labelings of a graph $G$ is called the $\Phi_k$-number and denoted by $\Phi_k(G)$.

For the special case that $k = 1$, the 1-labeling is indeed the conventional vertex coloring and we have $\Phi_1(G) = \chi(G) - 1$, where $\chi(G)$ is the chromatic number of $G$. Another special case is when $k = 2$, the 2-labeling is the same as the *distance two labeling* (or *$L(2, 1)$-labeling*) which has been studied extensively in the past years (cf. [1, 2, 3, 9, 10, 11, 12, 14]). The $\Phi_2$-number is known as the *$\lambda$-number* of $G$.

The radio $k$-labeling for large values of $k$ has also been investigated by several authors. Let $G$ be a connected graph. The maximum distance among all pairs of vertices in $G$ is the *diameter* of $G$, denoted by $\mathrm{diam}(G)$. The *radio labeling* (or *multi-level distance labeling*) is a radio $k$-labeling when $k = \mathrm{diam}(G)$. The $\Phi_{\mathrm{diam}(G)}$-number of $G$ is called the *radio number* of $G$, denoted by $\mathrm{rn}(G)$. The radio number for different families of graphs has been investigated in [6, 8, 15, 16, 17, 18, 19]. For instance, the radio number for paths and cycles has been studied in [6, 8, 19] and was recently settled in [18].

When $k = \mathrm{diam}(G) - 1$, a $k$-labeling is called an *antipodal labeling*. That is, an *antipodal labeling* (or *radio antipodal coloring*) for $G$ is a function, $f : V(G) \to \{0, 1, 2, \cdots\}$, such that the following is satisfied for any two vertices $u$ and $v$:

$$|f(u) - f(v)| \geq \mathrm{diam}(G) - d(u, v).$$

The *antipodal number* for $G$, denoted by $\mathrm{an}(G)$, is the minimum span of an antipodal labeling admitted by $G$. Notice that a radio labeling is a one-to-one function, while in an antipodal labeling, two vertices of distance $\mathrm{diam}(G)$ apart may receive the same label (this is where the name "antipodal" came from).

The antipodal labeling for graphs was first studied by Chartrand et al. [4, 5], in which, among other results, general bounds of an$(G)$ were obtained. Khennoufa and Togni [13] determined the exact value of an$(P_n)$ for paths $P_n$. The antipodal labeling for cycles $C_n$ was studied in [4], in which lower bounds for an$(C_n)$ were shown. In addition, the bound for the case $n \equiv 2 \pmod 4$ was proved to be the exact value of an$(C_n)$, and the bound for the case $n \equiv 1 \pmod 4$ was conjectured to be the exact value as well [4].

In this article, we confirm the conjecture mentioned above. Moreover, we determine the value of an$(C_n)$ for the case $n \equiv 3 \pmod 4$. For the case $n \equiv 0 \pmod 4$, we improve the known lower bound [4] and give an upper bound. It is conjectured that the upper bound is the exact value.

# 2 Lower Bounds

In this section, we establish lower bounds for an$(C_n)$. These bounds were proved by Chartrand et al [4]. We present here a different proof which includes techniques that will be used in later sections.

In an antipodal labeling, the number assigned to a vertex is called a *label*. Notice that as we are seeking for the minimum span of an antipodal labeling, without loss of generality we assume that the label 0 is used by any antipodal labeling. Consequently, the span of $f$ is the maximum label used.

In the following we introduce notations to be used throughout this article. Denote $V(C_n) = \{v_0, v_1, \cdots, v_{n-1}\}$, $v_i v_{i+1} \in E(C_n)$ for $0 \le i \le n-2$, and $v_{n-1}v_0 \in E(C_n)$. The diameter of $C_n$ is denoted by $d$, where $d = \lfloor n/2 \rfloor$. Every antipodal labeling $f$ for $C_n$ gives an ordering (which may not be unique) of the vertices according to the labels assigned . Denote the ordering by $(x_0, x_1, \cdots, x_{n-1})$, where $\{x_0, x_1, \cdots, x_{n-1}\} = V(C_n)$ and

$$0 = f(x_0) \le f(x_1) \le f(x_2) \le \cdots \le f(x_{n-1}).$$

Note, the span of $f$ is $f(x_{n-1})$.

For $i = 0, 1, \cdots, n-2$, we define the *distance gap* and *label gap*, respectively, by:

$$d_i = d(x_i, x_{i+1}), \quad f_i = f(x_{i+1}) - f(x_i).$$

By definition, it holds that $f_i \geq d - d_i$.

**Proposition 1** *For any three vertices $u$, $v$ and $w$ on a cycle $C_n$,*

$$d(u,v) + d(v,w) + d(u,w) \leq n.$$

**Proof.** Without loss of generality, assume $d(u,v), d(v,w) \leq d(u,w)$. If all the three vertices lie on one half of the cycle, then $d(u,v) + d(v,w) + d(u,w) = 2d(u,w) \leq n$. Otherwise, we have $d(u,v) + d(v,w) + d(u,w) = n$. □

**Lemma 2** *Let $f$ be an antipodal labeling for $C_n$, $n \geq 3$, with labels $f(x_0) \leq f(x_1) \leq \cdots \leq f(x_{n-1})$. Let $n = 4k + r$ for some $0 \leq r \leq 3$. Then for any $0 \leq i \leq n-3$,*

$$f(x_{i+2}) - f(x_i) = f_i + f_{i+1} \geq \begin{cases} k, & \text{if } r = 0, 1, 3; \\ k+1, & \text{if } r = 2. \end{cases}$$

**Proof.** By definition, we have $f(x_{i+1}) - f(x_i) \geq d - d(x_{i+1}, x_i)$, $f(x_{i+2}) - f(x_{i+1}) \geq d - d(x_{i+2}, x_{i+1})$, and $f(x_{i+2}) - f(x_i) \geq d - d(x_{i+2}, x_i)$. Summing up these three in-equalities and by Proposition 1, we get

$$\begin{aligned} 2(f(x_{i+2}) - f(x_i)) &\geq 3d - (d(x_i, x_{i+1}) + d(x_{i+1}, x_{i+2}) \\ &\qquad\qquad + d(x_i, x_{i+2})) \\ &\geq 3d - n. \end{aligned}$$

Therefore, $f_i + f_{i+1} = f(x_{i+2}) - f(x_i) \geq \lceil (3d - n)/2 \rceil$. The results then follow by immediate calculations for different values of $n$. □

**Corollary 3** [4] *Let $n = 4k + r$ for some $n \geq 3$ and $0 \leq r \leq 3$. Then*

$$\text{an}(C_n) \geq \begin{cases} k(2k-1), & \text{if } r = 0; \\ 2k^2, & \text{if } r = 1; \\ 2k(k+1), & \text{if } r = 2; \\ k(2k+1), & \text{if } r = 3. \end{cases}$$

**Proof.** Let $f$ be an antipodal labeling for $C_n$. The span of $f$ is

$$f(x_{n-1}) = f_0 + f_1 + \cdots + f_{n-2}.$$

By Lemma 2, the results follow by pairing up the terms in the above summation and leaving the last term $f_{n-2}$ (if $n$ is even) which is at least 0. □

In [4], it was proved that the equality in Corollary 3 holds for the case $n \equiv 2 \pmod 4$, and conjectured that the equality also holds for the case $n \equiv 1 \pmod 4$. This conjecture is confirmed in the next section.

# 3 $n = 4k + 1$

Let $f$ be an antipodal labeling for a cycle $C_n$ with $0 = f(x_0) \leq f(x_1) \leq \cdots \leq f(x_{n-1})$. In the rest of this article, we denote the permutation $\pi$ on $\{0, 1, 2, \cdots, n-1\}$ generated from $f$ with

$$x_i = v_{\pi(i)}.$$

For an integer $x$ and a positive integer $y$, we denote "$x$ mod $y$" as a binary operation which outputs an integer $z$ with $z \equiv x \pmod y$ and $0 \leq z \leq y - 1$.

In this section, we prove the following result:

**Theorem 4** *If $n = 4k + 1$ for some integer $k \geq 1$, then*

$$\mathrm{an}(C_n) = 2k^2.$$

**Proof.** By Corollary 3, it suffices to find an antipodal labeling with span $2k^2$. Two cases are considered. Recall $d = \mathrm{diam}(C_{4k+1}) = 2k$.

**Case 1. $k$ is odd** First, we label the $2k + 1$ vertices $x_0$, $x_2$, $\cdots$, $x_{4k}$ by

$$\pi(2i) = ki \bmod n, \text{ and } f(x_{2i}) = ki, \text{ for } i = 0, 1, 2, \cdots, 2k.$$

For instance, $\pi(2) = k$ (i.e., $x_2 = v_k$) and $f(x_2) = k$; and $\pi(4k) = 2k - \frac{k-1}{2}$ and $f(x_{4k}) = 2k^2$.

Secondly, we label the remaining vertices $x_1, x_3, \cdots, x_{4k-1}$ by $\pi(1) = \pi(4k) + k = 3k - \frac{k-1}{2}$; and $\pi(2i+1) = (\pi(2i-1) + k)$

mod $n$, for $i = 1, 2, \cdots, 2k-1$, with labels $f(x_{2i+1}) = (k-1)/2 + ki$ for $i = 0, 1, \cdots, 2k-1$. See Figure 1 for an example. In Figure 1 (and all other figures), the number inside the circle for each vertex is the label assigned to that vertex.

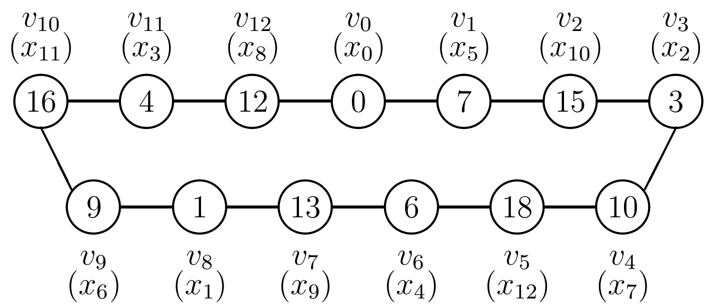


Figure 1: An antipodal labeling for $C_{13}$ with minimum span $\mathrm{an}(C_{13}) = 18$.

To see that $\pi$ is a permutation of $\{0, 1, \cdots, n-1\}$, we observe that $\pi(0), \pi(2), \cdots, \pi(4k), \pi(1), \pi(3), \cdots, \pi(4k-1)$ is a list of vertices winding around $C_n$ by jumping $k$ vertices between any two consecutive terms. Since $\gcd(n, k) = 1$, so $\pi$ is a permutation of $\{0, 1, \cdots, n-1\}$. In addition, one can easily check that for every $i$ the following hold:

$$f(x_{i+1}) - f(x_i) \geq d - d(x_{i+1}, x_i),$$

$$f(x_{i+2}) - f(x_i) = k = 2k - k = d - d(x_{i+2}, x_i),$$

$$f(x_{i+s}) - f(x_i) \geq 2k \geq d - d(x_{i+s}, x_i), \text{ for } s \geq 4.$$

Hence, to show that $f$ is an antipodal labeling, it suffices to verify $f(x_{i+3}) - f(x_i) \geq 2k - d(x_{i+3}, x_i)$. This is true since $d(x_{i+3}, x_i) = (k+1)/2$, and $f(x_{i+3}) - f(x_i) \in \{(3k-1)/2, (3k+1)/2\}$.

**Case 2. $k$ is even** Similar to Case 1, we first label the $2k+1$ vertices $x_0, x_2, \cdots, x_{4k}$, by $\pi(2i) = ki \bmod n$, for $i = 0, 1, \cdots, 2k$, using labels $f(x_{2i}) = ki$. Note that since $2k^2 \equiv n - \frac{k}{2} \pmod{n}$, we have $x_{4k} = v_{n-(k/2)}$.

Secondly, we label the remaining vertices by $\pi(1) = 2k+1$, $f(x_1) = 0$, and

$$\pi(2i+1) = \begin{cases} (\pi(2i-1)+k) & \text{mod } n, \quad \text{if } i \text{ is odd;} \\ (\pi(2i-1)+k+1) & \text{mod } n, \quad \text{if } i \text{ is even,} \end{cases}$$

with labels

$$f(x_{2i+1}) = \begin{cases} f(x_{2i-1})+k, & \text{if } i \text{ is odd;} \\ f(x_{2i-1})+k+1, & \text{if } i \text{ is even.} \end{cases}$$

See Figure 2 for an example.

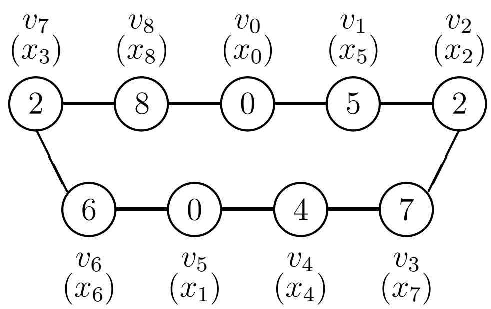


Figure 2: An antipodal labeling for $C_9$ with minimum span $\text{an}(C_9) = 8$.

By calculation, $\pi(1) = 2k+1 \equiv -2k \pmod{n}$, and for $1 \le i \le 2k-1$,

$$\pi(2i+1) \equiv \begin{cases} -ik & \pmod{n}, \quad \text{if } i \text{ is odd;} \\ -(i+2)k & \pmod{n}, \quad \text{if } i \text{ is even.} \end{cases}$$

Since $\pi(2i) = ki$ mod $n$, for $0 \le i \le 2k$, we conclude

$$\{\pi(i) : 0 \le i \le 4k\} = \{jk \text{ mod } n : -2k \le j \le 2k\}.$$

Since $\gcd(n, k) = 1$, $\pi$ is a permutation. Similar to Case 1, it is straightforward to check that $f$ is an antipodal labeling, and we shall leave the details to the reader. $\square$

# 4 $n = 4k + 3$

As it turned out (Theorem 5), the exact value of $\text{an}(C_{4k+3})$ is greater than the lower bound established in Corollary 3.

**Theorem 5** *For every integer $k \geq 0$,* $\text{an}(C_{4k+3}) = 2k^2 + 2k$.

Note, when $k = 0$ in Theorem 5, it is trivial that $\text{an}(C_3) = 0$. The following lemma will be used to prove Theorem 5 for $k \geq 1$.

**Lemma 6** *Let $f$ be an antipodal labeling for $C_n$ where $n = 4k + 3$, $k \geq 1$. If $f_i + f_{i+1} = k$ for some $0 \leq i \leq n - 3$, then the following hold:*

*(1)* $d(x_i, x_{i+2}) = k + 1$,

*(2)* $f_i = t$, $d_{i+1} = k + t + 1$, *and* $d_i = 2k - t + 1$, *for some* $t \in \{0, 1, \cdots, k\}$.

**Proof.** Recall $d = \text{diam}(C_{4k+3}) = 2k+1$. Assume $f_i + f_{i+1} = k$ for some $i$. By definition,

$$\begin{aligned} d(x_i, x_{i+2}) \geq d - (f(x_{i+2}) - f(x_i)) &= d - (f_{i+1} + f_i) \\ &= (2k + 1) - k = k + 1. \end{aligned}$$

On the other hand, by Proposition 1 and definition, we have

$$\begin{aligned} d(x_i, x_{i+2}) &\leq (4k + 3) - (d_i + d_{i+1}) \\ &\leq (4k + 3) - (d - f_i + d - f_{i+1}) \\ &= (4k + 3) - (4k + 2 - k) \\ &= k + 1. \end{aligned}$$

This verifies (1).

Let $f_i = t$ for some $t \in \{0, 1, \cdots, k\}$. By (1), the second equality in the above holds, which implies that $d_i = d - f_i$ and $d_{i+1} = d - f_{i+1}$. Therefore, (2) follows as $d = 2k + 1$. $\square$

**Lemma 7** *Let $f$ be an antipodal labeling for $C_n$ where $n = 4k+3$ for some integer $k \geq 1$. Then for any $0 \leq i \leq n - 5$,*

$$f_i + f_{i+1} + f_{i+2} + f_{i+3} \geq 2k + 1.$$

**Proof.** Assume to the contrary that for some $i$, $f_i + f_{i+1} + f_{i+2} + f_{i+3} \leq 2k$. By Lemma 2, $f_i + f_{i+1} = f_{i+2} + f_{i+3} = k$. By symmetry and by Lemma 6 (1), without loss of generality, assume $x_i = v_0$, $x_{i+2} = v_{k+1}$ and $x_{i+4} = v_{2(k+1)}$. By Lemma 6 (2), $f_i = t$ for some $0 \leq t \leq k$ and $d_i = 2k - t + 1$. Note, $x_{i+1} \neq v_{2k-t+1}$, for otherwise it would be $d_{i+1} = d(x_{i+1}, x_{i+2}) = k - t$, a contradiction. Hence, we conclude $x_{i+1} = v_{n-(2k-t+1)} = v_{2k+t+2}$. This implies $d(x_{i+4}, x_i) = t$. Because $f$ is an antipodal labeling, we have

$$\begin{aligned}
2k - t = f_{i+1} + f_{i+2} + f_{i+3} &= f(x_{i+4}) - f(x_{i+1}) \\
&\geq 2k + 1 - d(x_{i+4}, x_{i+1}) \\
&= 2k + 1 - t,
\end{aligned}$$

a contradiction. □

**Theorem 8** *For every integer* $k \geq 1$, $\text{an}(C_{4k+3}) \geq 2k^2 + 2k$.

**Proof.** By Lemmas 2 and 7, the span of an antipodal labeling $f$ for $C_{4k+3}$ has

$$\begin{aligned}
& f_0 + f_1 + \cdots + f_{4k+1} \\
= & \sum_{i=0}^{k-1} (f_{4i} + f_{4i+1} + f_{4i+2} + f_{4i+3}) + f_{4k} + f_{4k+1} \\
\geq & k(2k+1) + k = 2k^2 + 2k.
\end{aligned}$$

□

**Proof of Theorem 5.** For $k = 0$, $\text{an}(C_3) = 0$ is trivial as mentioned earlier. For $k \geq 1$, it remains to find an antipodal labeling for $C_{4k+3}$ with span equal to the desired number. First, we label the vertices $x_0, x_2, \cdots, x_{4k+2}$, by $\pi(0) = 0$ and $f(x_0) = 0$; and for $1 \leq i \leq 2k+1$,

$$\pi(2i) = \begin{cases} (\pi(2i-2) + k + 1) \mod n, & \text{if } i \text{ is odd;} \\ (\pi(2i-2) + k) \mod n, & \text{if } i \text{ is even,} \end{cases}$$

$$f(x_{2i}) = \begin{cases} f(x_{2i-2}) + k, & \text{if } i \text{ is odd;} \\ f(x_{2i-2}) + k + 1, & \text{if } i \text{ is even.} \end{cases}$$

Secondly, we label the remaining vertices by $\pi(1) = 2k + 2$ and $f(x_1) = 0$; and for $1 \leq i \leq 2k$,

$$\pi(2i+1) = (\pi(2i-1) + k + 1) \mod n, \text{ and } f(x_{2i+1}) = i(k+1).$$

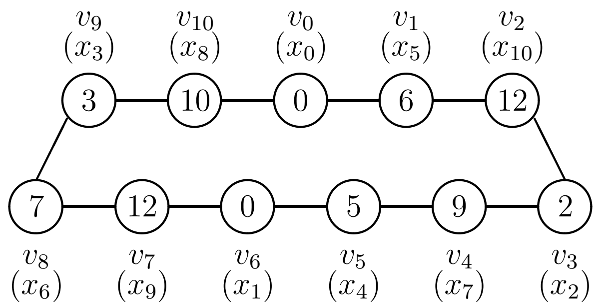


Figure 3: An antipodal labeling for $C_{11}$ with minimum span $\text{an}(C_{11}) = 12$.

See Figure 3 for an example.

By some calculation, one gets

$$\pi(2i+1) \equiv (i+2)(k+1) \pmod{n}, \text{ for } 0 \le i \le 2k, \text{ and}$$

$$\pi(2i) \equiv \begin{cases} -(i-2)(k+1) & \pmod{n}, \text{ if } i \text{ is odd;} \\ -i(k+1) & \pmod{n}, \text{ if } i \text{ is even,} \end{cases}$$

for $0 \le i \le 2k+1$. Hence, we conclude

$$\{\pi(i) : 0 \le i \le 4k+2\} = \{j(k+1) \bmod n : -2k \le j \le 2k+2\}.$$

Because $\gcd(n, k+1) = 1$, $\pi$ is a permutation. Similar to the proof of Theorem 4, it is straightforward to show that $f$ is an antipodal labeling, and we shall leave the details to the reader. This completes the proof of Theorem 5. □

## 5 $n = 4k$

Note, it is trivial that $\text{an}(C_4) = 1$. For cycles with $n = 4k$ nodes, $k \ge 2$, we improve the lower bound in Corollary 3 and give an upper bound.

**Theorem 9** *For every integer $k \geq 2$,*

$$2k^2 - \lfloor k/2 \rfloor \leq \text{an}(C_{4k}) \leq 2k^2 - 1.$$

The following lemma will be used to prove the lower bound for $\text{an}(C_{4k})$ in Theorem 9. Recall that $d = \text{diam}(C_{4k}) = 2k$.

**Lemma 10** *Let $f$ be an antipodal labeling of $C_{4k}$, for some integer $k \geq 2$. If $f_i + f_{i+1} = k$ for some $0 \leq i \leq n-3$, then $d(x_i, x_{i+2}) = k$.*

**Proof.** Assume $f_i + f_{i+1} = k$ for some $0 \leq i \leq n-3$. Then $d(x_i, x_{i+2}) \geq d - (f_i + f_{i+1}) = k$. On the other hand, by Proposition 1 and definition,

$$\begin{aligned} d(x_i, x_{i+2}) &\leq n - (d_i + d_{i+1}) \\ &\leq 4k - (d - f_i + d - f_{i+1}) \\ &= k. \end{aligned}$$

□

**Lemma 11** *Let $f$ be an antipodal labeling of $C_{4k}$, $k \geq 2$. Then for any $0 \leq i \leq n-9$,*

$$\sum_{j=0}^{7} f_{i+j} \geq 4k + 1.$$

**Proof.** Assume to the contrary, for some $0 \leq i \leq n-9$, we have $\sum_{j=0}^{7} f_{i+j} \leq 4k$. By Lemma 2, $f_i + f_{i+1} = f_{i+2} + f_{i+3} = f_{i+4} + f_{i+5} = f_{i+6} + f_{i+7} = k$. By Lemma 10, $d(x_i, x_{i+2}) = d(x_{i+2}, x_{i+4}) = d(x_{i+4}, x_{i+6}) = d(x_{i+6}, x_{i+8}) = k$. Since $n = 4k$, it is impossible that all these four equations hold. So the result follows. □

**Lemma 12** *Let $f$ be an antipodal labeling of $C_{4k}$, $k \geq 2$. The following are true.*

*(1) If $f_i + f_{i+1} = k$ for some $0 \leq i \leq n-4$, then $f_{i+2} \geq f_i$.*

*(2) If $f_i + f_{i+1} = k$ for some $1 \leq i \leq n-3$, then $f_{i-1} \geq f_{i+1}$.*

(3) *If* $f_i + f_{i+1} + f_{i+2} + f_{i+3} = 2k$ *for some* $0 \le i \le n-6$*, then* $f_{i+4} \ge f_i \ge 1$.

(4) *If* $\sum_{j=0}^{7} f_{i+j} = 4k+1$ *for some* $0 \le i \le n-10$*, then* $f_{i+8} \ge f_i$.

(5) *If* $\sum_{j=0}^{7} f_{i+j} = 4k+1$ *for some* $0 \le i \le n-10$*, then* $f_{i+8} \ge 1$.

(6) *For any* $0 \le i \le n-6$*,* $\sum_{j=0}^{4} f_{i+j} \ge 2k+1$.

(7) *For any* $0 \le i \le n-10$*,* $\sum_{j=0}^{8} f_{i+j} \ge 4k+2$.

**Proof.** To prove (1), assume $f_i + f_{i+1} = k$ for some $0 \le i \le n-4$. By Lemma 2, $f_{i+2} + f_{i+1} \ge k = f_{i+1} + f_i$, hence $f_{i+2} \ge f_i$. (2) follows by a similar argument.

To prove (3), assume $f_i + f_{i+1} + f_{i+2} + f_{i+3} = 2k$ for some $0 \le i \le n-6$. Then by Lemma 2, $f_i + f_{i+1} = f_{i+2} + f_{i+3} = k$. By Lemma 10, $d(x_i, x_{i+2}) = d(x_{i+2}, x_{i+4}) = k$, so $d(x_i, x_{i+4}) = 2k$. This implies that $d_i < 2k$, as $n = 4k$. By definition of antipodal labeling, $f_i \ge 1$. Hence, by (1), we have $f_{i+4} \ge f_{i+2} \ge f_i \ge 1$.

To prove (4), assume $\sum_{j=0}^{7} f_{i+j} = 4k+1$ for some $0 \le i \le n-10$. By Lemma 11, $\sum_{j=1}^{8} f_{i+j} \ge 4k+1 = \sum_{j=0}^{7} f_{i+j}$, hence $f_{i+8} \ge f_i$.

To prove (5), assume $\sum_{j=0}^{7} f_{i+j} = 4k+1$ for some $0 \le i \le n-10$. By Lemma 2, we have $f_i + f_{i+1} = f_{i+2} + f_{i+3} = k$ or $f_{i+4} + f_{i+5} = f_{i+6} + f_{i+7} = k$. For the former case, the result follows by (4) and (3); for the latter case, the results follows by (3).

(6) follows by (3) and Lemma 2; and (7) follows by (5) and Lemma 11. □

**Corollary 13** *For any integer* $k \ge 2$*,* $\mathrm{an}(C_{4k}) \ge 2k^2 - \lfloor k/2 \rfloor$.

**Proof.** For $k=2$, by Lemma 2 and Lemma 12 (6), the span of an antipodal labeling $f$ for $C_8$ has $f(x_7)=(f_0+f_1+\cdots+f_4)+(f_5+f_6)\geq 5+2=2k^2-\lfloor k/2\rfloor$.

For $k\geq 3$, by Lemmas 2, 11 and 12 (7), the span of an antipodal labeling $f$ for $C_{4k}$ has

$$\begin{aligned} f(x_{4k-1}) &= \sum_{i=0}^{8} f_i + \sum_{i=1}^{\lfloor (4k-9)/8\rfloor}\left(\sum_{j=1}^{8}(f_{8i+j})\right) \\ &\qquad + f_{8\lfloor(4k-1)/8\rfloor+1} + f_{8\lfloor(4k-1)/8\rfloor+2} + \cdots + f_{4k-2} \\ &\geq \begin{cases} (4k+2)+[2k^2-(11/2)k-3/2]+k, & k \text{ is odd} \\ (4k+2)+[2k^2-(15/2)k-2]+3k, & k \text{ is even} \end{cases} \\ &= 2k^2-\lfloor k/2\rfloor. \end{aligned}$$

□

**Proof of Theorem 9.** It remains to find an antipodal labeling for $C_{4k}$ with span $2k^2-1$. First, we label the vertices $x_0$, $x_2$, $\cdots$, $x_{4k-2}$, by $\pi(0)=0$ and $f(x_0)=0$; and for $1\leq i\leq 2k-1$,

$$\pi(2i)=\begin{cases} (\pi(2i-2)+k) \mod n, & \text{if } i \text{ is odd;} \\ (\pi(2i-2)+k+1) \mod n, & \text{if } i \text{ is even,} \end{cases}$$

$$f(x_{2i})=\begin{cases} f(x_{2i-2})+k, & \text{if } i \text{ is odd;} \\ f(x_{2i-2})+k+1, & \text{if } i \text{ is even.} \end{cases}$$

Secondly, we label the remaining vertices by: For $0\leq i\leq 2k-1$,

$$\pi(2i+1)=(\pi(2i)+2k) \mod n, \text{ and } f(x_{2i+1})=f(x_{2i}).$$

See Figure 4 for an example.

By calculation, one gets the following for $0\leq i\leq k-1$:

$$\begin{aligned} \pi(4i) &\equiv i(2k+1) \pmod n. \\ \pi(4i+1) &\equiv (i+2k)(2k+1) \pmod n. \\ \pi(4i+2) &\equiv \begin{cases} (i+3k)(2k+1) \pmod n, & \text{if } k \text{ is odd;} \\ (i+k)(2k+1) \pmod n, & \text{if } k \text{ is even.} \end{cases} \\ \pi(4i+3) &\equiv \begin{cases} (i+k)(2k+1) \pmod n, & \text{if } k \text{ is odd;} \\ (i+3k)(2k+1) \pmod n, & \text{if } k \text{ is even.} \end{cases} \end{aligned}$$

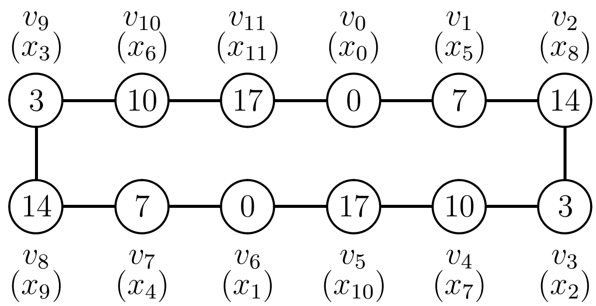


Figure 4: An antipodal labeling for $C_{12}$ with minimum span $\mathrm{an}(C_{12}) = 17$.

Therefore, we conclude

$$\{\pi(i) : 0 \leq i \leq 4k-1\} = \{j(2k+1) \bmod n : 0 \leq j \leq 4k-1\}.$$

Because $\gcd(n, 2k+1) = 1$, $\pi$ is a permutation. Similar to the proof of Theorem 4, it is straightforward to show that $f$ is an antipodal labeling, and we shall leave the details to the reader. This completes the proof of Theorem 9. □

We conjecture that $\mathrm{an}(C_{4k})$ is equal to the upper bound in Theorem 9.

**Conjecture 1** *For any* $k \geq 1$, $\mathrm{an}(C_{4k}) = 2k^2 - 1$.

A case analysis has confirmed the above conjecture for $k \leq 5$.

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