Asymptotic Clique Covering Ratios of Distance Graphs

Daphne D.-F. Liu *
Department of Mathematics and Computer Science
California State University, Los Angeles
Los Angeles, CA 90032
Emil: dliu@calstatela.edu

Xuding Zhu †
Department of Applied Mathematics
National Sun Yat-sen University
Kaoshing 80424, Taiwan
Email: zhu@ibm28.math.nsysu.edu.tw

June 1, 1999 (Revised: July 31, 2001)

Abstract

Given a finite set D of positive integers, the distance graph $G(Z, D)$ has Z as the vertex set and $\{ij : |i-j| \in D\}$ as the edge set. Given D, the asymptotic clique covering ratio is defined as $S(D) = \limsup_{n \to \infty} \frac{n}{cl(n)}$, where $cl(n)$ is the minimum number of cliques covering any consecutive n vertices of $G(Z, D)$. The parameter $S(D)$ is closely related to the ratio $\frac{\mathrm{sp_T(G)}}{\chi(G)}$ of a graph G, where $\chi(G)$ and $\mathrm{sp_T(G)}$ denote, respectively, the chromatic number and the optimal span of a T-coloring of G. We prove that for any finite set D, $S(D)$ is a rational number and can be realized by a "periodical" clique covering of $G(Z, D)$. Then we investigate the problem for which sets D the equality $S(D) = \omega(G(Z, D))$ holds. (In general, $S(D) \leq \omega(G(Z, D))$, where $\omega(G)$ is the clique number of G.) This problem turns out to be related to T-colorings and to fractional chromatic number and circular chromatic number of distance graphs. Through such connections, we shall show that the equality $S(D) = \omega(G(Z, D))$ holds for many classes of distance graphs. Moreover, we raise questions regarding other such connections.

*Supported in part by the National Science Foundation under grant DMS 9805945.
†Supported in part by the National Science Council, R. O. C., under grant NSC87-2115-M110-004.

Keywords: Distance graph, T-coloring, T-span, clique covering, chromatic number, periodical clique covering

1 Introduction

Let D be a set of positive integers, the *distance graph* generated by D, denoted by $G(Z, D)$, has all the integers Z as the vertex set, and two vertices are adjacent if their absolute difference falls within the set D. The set D is called the *distance set* (or D-set for short) of the graph $G(Z, D)$. For $n \geq 1$, we denote by $G(n, D)$ the subgraph of $G(Z, D)$ induced by the set of vertices $\{0, 1, \cdots, n-1\}$. The study of the chromatic number of distance graphs was initiated by Eggleton, Erdős and Skilton [10]. Their motivation was to study the 1-dimensional analogous of the well-known plane coloring problem (i.e., finding the minimum number of colors needed to color the plane so that no two points of unit distance are colored the same color). Later on, it was found that the chromatic number and fractional chromatic number of distance graphs are related to many other problems, such as T-colorings [3, 24], diophantine approximations [35], density of D-sets [15] and circulant graphs [20], etc.

Focusing on finite distance sets D, we consider the problem of covering the vertices of distance graphs $G(Z, D)$ by cliques. This problem is equivalent to proper vertex-coloring the complement of $G(Z, D)$, which is also a distance graph whose distance set is $Z^+ - D$. In this sense, we are still considering vertex-coloring problem for distance graphs. We choose to use the language of clique-covering instead of vertex-coloring because it is easier to deal with distance graphs whose distance sets are finite. A useful observation in coloring distance graphs is that we only need to color the subgraph of $G(Z, D)$ induced by the set of non-negative integers $\{0, 1, \cdots\}$. Therefore, throughout the article (especially in Sections 2 and 3), unless indicated, we shall restrict our discussion of clique coverings to this subgraph.

Our study of clique covering of distance graphs with finite distance sets is motivated by problems concerning T-colorings which arose from the channel assignment problem introduced by Hale [16]. Given a finite set T (called T-set) of non-negative integers with $0 \in T$, a *T-coloring* of a graph $G = (V, E)$ is a mapping $\phi : V \to \{0, 1, 2, \cdots\}$ such that if $xy \in E(G)$, then $|\phi(x) - \phi(y)| \notin T$. The *span* of a T-coloring ϕ of G is defined as $\mathrm{sp_{T,\phi}(G)} = \mathrm{max}\phi(\mathrm{V}) - \mathrm{min}\phi(\mathrm{V})$. The *$T$-span* of G denoted by $\mathrm{sp_T(G)}$ is defined as $\min\{\mathrm{sp_{T,\phi}(G)} : \phi \text{ is a } T\text{-coloring of } G\}$.

Given a D-set, denote the complement of $G(Z, D)$ by $\overline{G}(Z, D)$ (similarly for $\overline{G}(n, D)$). A *homomorphism* from a graph $G = (V, E)$ to another graph $H = (V', E')$ is a mapping $h : V \to V'$ such that $h(x)h(y)$ is an edge of H whenever xy is an edge of G. Let $D = T - \{0\}$, a T-coloring of G is a homomorphism from G to $\overline{G}(Z, D)$ and vice versa. Thus, an equivalent definition of $\mathrm{sp_T(G)}$ is

$$\mathrm{sp_T(G)} = \min\{\mathrm{n} - 1 : \mathrm{G \text{ admits a homomorphism to }} \overline{G}(n, D)\ \} \text{ where } D = T - \{0\}.$$

It is usually very difficult to determine the minimum span $\mathrm{sp_T(G)}$ of a graph G. Much of the earlier efforts in the study of T-coloring have been focused on finding upper and lower bounds of $\mathrm{sp_T(G)}$ in terms of other parameters such as $\omega(G)$, the clique number (i.e., the size of a maximum clique in G), and $\chi(G)$, the chromatic number of G [5, 28, 21, 22, 27]. For a given T-set, let σ_n denote the minimum span of K_n. It is easy to see that $\sigma_{\omega(G)} \le \mathrm{sp_T(G)} \le \sigma_{\chi(\mathrm{G})}$ [5].

We are interested in the ratio $\frac{\mathrm{sp_T(G)}}{\chi(G)}$. The two parameters $\mathrm{sp_T(G)}$ and $\chi(G)$ are certainly closely related to each other. For instance, when $\chi(G)$ goes to infinity, then so does $\mathrm{sp_T(G)}$. We are interested in finding quantitative relations between these two parameters. In particular, we shall investigate the range of the ratio $\frac{\mathrm{sp_T(G)}}{\chi(G)}$.

An upper bound of $\frac{\mathrm{sp_T(G)}}{\chi(G)}$ can be obtained easily as follows. Because $\mathrm{sp_T(G)} \le \sigma_{\chi(\mathrm{G})}$ for any graph G, we have

$$\frac{\mathrm{sp_T(G)}}{\chi(G)} \le \frac{\sigma_{\chi(G)}}{\chi(G)}.$$

In the case that $G = K_n$, then $\frac{\mathrm{sp_T(G)}}{\chi(G)} = \frac{\sigma_{\chi(G)}}{\chi(G)} = \frac{\sigma_n}{n}$. So the upper bound for $\frac{\mathrm{sp_T(G)}}{\chi(G)}$

above is sharp. Note that $\frac{\sigma_n}{n}$ is bounded, since $\sigma_n \leq n \times (\max\{d : d \in D\} + 1)$.

The asymptotic ratio $R(T) = \lim_{n\to\infty} \frac{\sigma_n}{n}$ has been studied by several authors. It was proved independently and differently by Rabinowitz and Proulx [26] and by Griggs and Liu [15] that for any given finite T-set, $R(T)$ exists and is a rational number ≥ 2, except when $T = \{0\}$, $R(T) = 1$. Moreover, the difference sequence $\{\sigma_{n+1} - \sigma_n\}_{n=1}^{\infty}$ is eventually periodic [15]. It was noted in [15] that $R(T)$ is equivalent to the reciprocal of "density of sequences with missing distances," an earlier number theory problem studied by Cantor and Gordon [1] and by Haralambis [17].

The study of the clique covering of the distance graphs arises from the approach of the lower bound of the ratio $\frac{\mathrm{sp_T(G)}}{\chi(G)}$. Suppose $\mathrm{sp_T(G) = n - 1}$, then G admits a homomorphism to $\overline{G}(n, D)$, where $D = T - \{0\}$. Note that a proper k-coloring of a graph G is simply a homomorphism of G to K_k. Since homomorphism (considered as a binary relation on the set of graphs) is transitive (i.e., if H admits a homomorphism to H' and H' admits a homomorphism to H'' then H admits a homomorphism to H''), we know that $\chi(G) \leq \chi(\overline{G}(n, D))$. Therefore

$$\frac{\mathrm{sp_T(G)}}{\chi(G)} \geq \frac{n-1}{\chi(\overline{G}(n, D))}.$$

By taking $G = \overline{G}(n, D)$ such that $\overline{G}(n, D)$ is a core (i.e., $\overline{G}(n, D)$ does not admit a homomorphism to any of its proper subgraphs, for example, we may choose $n = \mathrm{sp_T(K_m)} + 1$ for some integer m), we know that the lower bound for the ratio $\frac{\mathrm{sp_T(G)}}{\chi(G)}$ above is also sharp.

Similarly to the study of the parameter $R(T)$, we investigate the asymptotic ratio $\limsup_{n\to\infty} \frac{n}{\chi(\overline{G}(n,D))}$. (As n goes to infinity, we may ignore the minus 1 in the numerator.) The *clique covering number* $cl(G)$ of a graph G is the minimum number k such that there exist k cliques in G that cover all vertices of G. It is easy to see that $\chi(\overline{G}) = cl(G)$ holds for any graph G. This leads to the definition of the *asymptotic* *clique covering ratio* $S(D)$ of a distance graph $G(Z, D)$:

$$S(D) = \limsup_{n\to\infty} \frac{n}{cl(G(n, D))}.$$

In this article, we shall prove that for any given finite D-set, $S(D)$ is a rational number. Moreover, $S(D)$ can be obtained by a "periodical" pattern of clique covering of $G(Z, D)$.

Intuitively, $S(D)$ is the average size of the cliques in an optimal clique covering of $G(Z, D)$, and it measures how efficiently one can cover the vertices of $G(Z, D)$ by cliques, i.e., using as few cliques as possible. Hence a trivial upper bound for $S(D)$ is

$$S(D) \leq \omega(G(Z, D)).$$

The class of distance graphs such that $S(D) = \omega(G(Z, D))$ is related to many other well studied classes of graphs. In the study of T-colorings, those T-sets for which $\mathrm{sp_T(G)} = \mathrm{sp_T(K_{\chi(G)})}$ for every graph G were investigated by several authors [5, 27, 21, 22, 24]. We shall show that if T is such a T-set, then $S(D) = \omega(G(Z, D))$, where $D = T - \{0\}$. We shall also give examples of sets D for which $S(D) < \omega(G(Z, D))$ as well as sets D for which $S(D) = \omega(G(Z, D))$ but $\mathrm{sp_T(G)} \neq \mathrm{sp_T(K_{\chi(G)})}$ for some graph G, where $T = D \cup \{0\}$.

If $S(D) = \omega(G(Z, D))$, then the vertices of $G(Z, D)$ can be partitioned into cliques of maximum size. This suggests that the complement $\overline{G}(Z, D)$ of the distance graph $G(Z, D)$ resembles a perfect graph. We are thus led to ask the questions regarding the relations between the two classes of distance graphs, namely, distance graphs $G(Z, D)$ for which $\chi(G(Z, D)) = \omega(G(Z, D))$, and distance graphs $G(Z, D)$ for which $S(D) = \omega(G(Z, D))$. In particular, whether or not one of the classes is a subset of the other. We shall show that there are distance graphs for which $S(D) = \omega(G(Z, D))$ but $\chi(G(Z, D)) \neq \omega(G(Z, D))$. However, we suspect that $\chi(G(Z, D)) = \omega(G(Z, D))$ may imply $S(D) = \omega(G(Z, D))$. By discussing regular colorings of distance graphs, we provide some support for this suspicion.

2 Definitions and preliminary results

In the remaining part of this article, unless indicated, let D be a fixed finite set of positive integers, and let d be the maximum element of D. For simplicity, we shall denote the distance graph $G(Z, D)$ by G.

Let f be a clique covering of G. We may regard f as a mapping that assigns to each vertex x a color $f(x)$. Since the vertices of the same color induce a clique in G, the number of cliques covering a subset $S \subset Z$ by f is $|f(S)|$. For $i < j$, we shall denote by $f[i, j]$ the subsequence $(f(i), f(i+1), \cdots, f(j))$, denote by $f\{i, j\}$ the set $\{a : a = f(\ell), i \leq \ell \leq j\}$. Let $f[i, .]$ denote the terminal segment of f starting at i, let $f[., i]$ denote the initial segment of f ending at i, and let $f[., .]$ denote the whole sequence f. The notations $f\{i, .\}, f\{., i\}$ and $f\{., .\}$ are defined analogously.

Definition 1 *A* clique covering sequence *of G is an infinite sequence of integers $f = (f(0), f(1) \cdots)$ such that $f(i) = f(j)$ implies that $|i - j| \in D$. A* partial clique covering sequence *of G is a finite sequence $f = (f(a), f(a+1), \cdots, f(k))$ such that $f(i) = f(j)$ implies that $|i - j| \in D$.*

The following lemma follows directly from the definition.

Lemma 1 *An infinite sequence f is a clique covering sequence of G if and only if $f(i) \neq f(j)$ when $i \leq j - d - 1$, and for any i, $f[i, i+d]$ is a partial clique covering sequence of G.*

Definition 2 *If $f[i, j]$, $i < j$, is a partial clique covering sequence of G, then the covering ratio of $f[i, j]$ is defined as*

$$S_{f[i,j]} = \frac{j - i + 1}{|f\{i, j\}|}.$$

If $f = (f(0)f(1) \cdots)$ is a clique covering sequence of G, the asymptotic covering ratio *of f, denoted by S_f, is defined as*

$$S_f = \limsup_{n \to \infty} \frac{n}{|f\{0, n-1\}|}.$$

Two clique covering sequences f and f' are *isomorphic* if there is a one-to-one mapping $\sigma : f\{.,.\} \to f'\{.,.\}$ such that for all x, $f'(x) = \sigma(f(x))$. Two partial clique covering sequences $f[i, i+k]$ and $f'[j, j+k']$ are isomorphic, denoted by $f[i, i+k] \equiv f'[j, j+k']$, if they have the same length (i.e., $k = k'$) and there is a one-to-one onto mapping $\sigma : f\{i, i+k\} \to f'\{j, j+k'\}$ such that for all $0 \le x \le k$, $f'(j+x) = \sigma(f(i+x))$. We call f a *periodical clique covering* if there exists an integer $p \ge 2d$ such that $f[0, .]$ and $f[p, .]$ are isomorphic.

Definition 3 *Suppose f is a clique covering sequence or a partial clique covering sequence such that $f[i, i+2d] \equiv f[j, j+2d]$. Then $f[i, j-1]$ is called a* complete segment *of f. The* adjusted covering ratio *of a complete segment $f[i, j-1]$ is defined as*

$$AS_{f[i,j-1]} = \frac{j-i}{|f\{i, j-1\} - f\{j, j+2d\}|}.$$

Note that the adjusted covering ratio of a complete segment $f[i, j-1]$ is no less than the covering ratio of the subsequence $f[i, j-1]$, i.e., $S_{f[i,j-1]} \le AS_{f[i,j-1]}$.

Definition 4 *Suppose $f = (f(0)f(1)\cdots f(k))$ is a partial clique covering sequence of G and $f[i, j-1]$ is a complete segment of f. Let $\sigma : f\{j, j+2d\} \to f\{i, i+2d\}$ be a mapping such that $\sigma(f(j+t)) = f(i+t)$ for $t = 0, 1, \cdots, 2d$. Let $f' = (f'(0), f'(1), \cdots, f'(k-j+i))$ be the sequence defined by:*

$$f'(x) = \begin{cases} f(x), & \text{if } x \le i; \\ \sigma(f(x+j-i)), & \text{if } x \ge i+1 \text{ and } f(x+j-i) \in f\{j, j+2d\}; \\ f(x+j-i), & \text{if } x \ge i+1 \text{ and } f(x+j-i) \notin f\{j, j+2d\}. \end{cases}$$

Then f' is called the sequence obtained from f by cutting off *the complete segment $f[i, j-1]$.*

Lemma 2 *Suppose $f = (f(0), f(1), \cdots f(k))$ is a partial clique covering sequence of G and $f[i, j-1]$ is a complete segment of f, $k \ge j+2d$. Then the sequence f'*

obtained from f by cutting off $f[i, j-1]$ is still a partial clique covering sequence of G. Moreover,

$$|f'\{0, k-(j-i)\}| = |f\{0, k\}| - |f\{i, j-1\} - f\{j, j+2d\}|.$$

Proof. It follows from the definition of f' that $f'[0, i+2d] \equiv f[0, i+2d]$, and $f'[i, k-(j-i)] \equiv f[j, k]$. Therefore for any $0 \leq t \leq k-(j-i)$, $f'[t, t+d] \equiv f[t+j-i, t+d+j-i]$ which is a partial clique covering of G.

By Lemma 1, it remains to show that if $f'(x) = f'(y)$ and $x < y$, then $x \geq y-d$. Assume to the contrary that there exist x and y, $x < y-d$, such that $f'(x) = f'(y)$. Then $x \leq i-1$ and $y \geq i+2d+1$, as $f'[0, i+2d] \equiv f[0, i+2d]$, $f'[i, k-(j-i)] \equiv f[j, k]$ and f is a partial clique covering sequence. This implies that $f'(x) = f(x) \in f\{0, i-1\}$. Moreover, either $f'(y) = f(y+j-i) \in f\{j+2d+1, .\}$; or $f'(y) = \sigma(f(y+j-i)) = \sigma(f(j+t)) = f(i+t) = f(x)$ where $f(y+j-i) = f(j+t)$ for some $0 \leq t \leq 2d$. In the former case, $f'(x) = f'(y) \in f\{0, i-1\} \cap f\{j+2d+1, .\}$, contrary to Lemma 1, as f is a clique covering sequence. In the latter case, $j+t$ is adjacent to $y+j-i$, and x is adjacent to $i+t$, so $j+t \geq y+j-i-d \geq j+d+1$ and $x \geq i+t-d$. Then we get a contradiction that $t \geq d+1$ and $t \leq d$. Hence, f' is a partial clique covering sequence of G.

Now we prove the second part. It follows from the definition that exactly those colors in the set $f\{i, j+2d\} - f\{i, i+2d\}$ which are used by f but not by f'. Because $|f\{i, i+2d\}| = |f\{j, j+2d\}|$, so $|f\{i, j+2d\} - f\{i, i+2d\}| = |f\{i, j+2d\} - f\{j, j+2d\}| = |f\{i, j-1\} - f\{j, j+2d\}|$. Hence the moreover part follows. Q.E.D.

Corollary 3 *Suppose $f = (f(0), f(1), \cdots, f(q+2d))$ is a partial clique covering with $f[i, j-1]$ a complete sequence, where $i \geq 0$ and $j \leq q$. Let f' be the partial clique covering obtained from f by cutting off $f[i, j-1]$. If $AS_{f[i,j-1]} \leq S_{f[0,q+2d]}$, then $S_{f'[0,q+2d-(j-i)-1]} \geq S_{f[0,q+2d]}$. Moreover, if $f[0, 2d] \equiv f[q, q+2d]$ and $AS_{f[i,j-1]} \leq AS_{f[0,q-1]}$, then $AS_{f'[0,q-(j-i)-1]} \geq AS_{f[0,q-1]}$.*

Proof. By Lemma 2, $|f'\{0,q+2d-(j-i)\}| = |f\{0,q+2d\}|-|f\{i,j-1\}-f\{j,j+2d\}|$. Hence, we have

$$\begin{aligned} S_{f'[0,q+2d-(j-i)]} &= \frac{q+2d-(j-i)+1}{|f'\{0,q+2d-(j-i)\}|} \\ &= \frac{q+2d-(j-i)+1}{|f\{0,q+2d\}|-|f\{i,j-1\}-f\{j,j+2d\}|} \\ &\geq \frac{q+2d+1}{|f\{0,q+2d\}|} \\ &= S_{f[0,q+2d]}. \end{aligned}$$

The inequality above follows from the assumption that

$$S_{f[0,q+2d]} = \frac{q+2d+1}{|f\{0,q+2d\}|} \geq \frac{j-i}{|f\{i,j-1\}-f\{j,j+2d\}|} = AS_{f[i,j-1]},$$

and the fact that $a/b \geq c/d$ implies that $(a+c)/(b+d) \leq a/b$.

The moreover part can be proved similarly. We shall leave it to the reader. Q.E.D.

Definition 5 *Suppose $g_0 = (g(0), g(1), \cdots, g(q+2d))$ is a partial clique covering sequence of G, where $q \geq 1$, and $g_0[0,2d] \equiv g_0[q,q+2d]$, i.e., $g_0[0,q-1]$ is a complete segment of g_0. Let $g_1, g_2, \cdots,$ be partial clique covering sequences of G isomorphic to g_0 such that $g_{i+1}[0,2d] = g_i[q,q+2d]$ and $(g_i\{0,q+2d\} - g_i\{0,2d\}) \cap g_j\{0,q+2d\} = \emptyset$ for all $j < i$. For any integer x, let $x = i_x q + a_x$, where $0 \leq a_x \leq q-1$. Then the sequence f defined by $f(x) = g_{i_x}(a_x)$ is called a sequence obtained by tiling g.*

Lemma 4 *Suppose $g = (g(0), g(1), \cdots, g(q+2d))$ is a partial clique covering sequence of G, where $q \geq 1$ and $g[0,2d] \equiv g[q,q+2d]$. If f is obtained by tiling g, then f is a periodical clique covering sequence of G with period q. Moreover, $S_f = AS_{g[0,q-1]}$.*

Proof. For any $t \geq 0$, $f[t,t+d]$ is a subsequence of $g_i[0,q+2d]$ for some i. Hence $f[t,t+d]$ is a partial clique covering sequence of G. By Lemma 1, it remains to show that $f(x) \neq f(y)$ when $x < y-d$. Assume to the contrary that there exist x, y such that $f(x) = f(y)$ and $x < y - d$. Let x, y be such a pair that $y - x$ is minimum. By definition, $f[x, y]$ is not a subsequence of $g_i[0, q+2d]$ for any i. Thus $y > i_x q + q + 2d$, $x < i_y q$, $i_x < i_y$, and $f(x) = g_{i_x}(a_x) = f(y) = g_{i_y}(a_y)$. By definition of g_j, we know that $g_{i_x}\{0, q + 2d\} \cap g_{i_y}\{0, q + 2d\} \subset g_{i_x}\{q, q + 2d\} \cap g_{i_y}\{0, 2d\}$. Therefore, there exists some t, $q \leq t \leq q + 2d$, such that $g_{i_x}(t) = g_{i_x}(a_x)$. Since g_{i_x} is a partial clique covering sequence, $t - a_x \leq d$, hence $t < q + d$. Then $f(i_x q + t) = f(x) = f(y)$, and $y - (i_x q + t) > i_x q + q + 2d - i_x q - q - d = d$. This contradicts the minimality of $y - x$.

It follows from the definition that $|f\{0, kq + 2d\}| = k|g\{0, q - 1\} - g\{q, q + 2d\}| + |g\{q, q + 2d\}|$. Therefore

$$\begin{aligned} S_f &= \lim_{k\to\infty} \frac{kq + 2d + 1}{|f\{0, kq + 2d\}|} \\ &= \lim_{k\to\infty} \frac{kq + 2d + 1}{k|g\{0, q - 1\} - g\{q, q + 2d\}| + |g\{q, q + 2d\}|} \\ &= \frac{q}{|g\{0, q - 1\} - g\{q, q + 2d\}|} \\ &= AS_{g[0,q-1]}. \end{aligned}$$

Q.E.D.

3 Periodical clique covering sequence with optimal covering ratio

In this section, we prove the following result:

Theorem 5 *Given D, there exists a periodical clique covering sequence f with $S_f = S(D)$.*

Proof. Let $\ell = (2d+1)^{2d+2}$, and let $\mathcal{Q}$ be the set of all non-isomorphic partial clique covering sequences $g = (g(0), g(1), \cdots, g(q+2d))$ of G such that $q \leq \ell$ and $g[0, 2d] \equiv g[q, q+2d]$. By Lemma 4, it suffices to prove that there exists a partial clique covering sequence $g = (g(0), g(1), \cdots, g(q+2d))$ in $\mathcal{Q}$ such that $AS_{g[0,q-1]} = S(D)$. Assume to the contrary that for any partial clique covering sequence g in $\mathcal{Q}$, $AS_{g[0,q-1]} < S(D)$. Since the set $\mathcal{Q}$ is finite, there exists an $\epsilon > 0$ such that $AS_{g[0,q-1]} < S(D) - \epsilon$ for every $g \in \mathcal{Q}$.

Claim. There exists a partial clique covering sequence $g = (g(0), g(1), \cdots, g(q+2d))$ such that $g[0, 2d] \equiv g[q, q+2d]$ and $AS_{g[0,q-1]} > S(D) - \epsilon$.

Proof. By definition of $S(D)$ there exists a clique covering sequence f of G and an integer n' for which $S_{f[0,n'-1]} = \frac{n'}{|f\{0,n'-1\}|} > S(D) - \epsilon$. Suppose $|f\{0, 2d\}| = m$. Let $c_1, c_2, \cdots, c_m$ be m new colors not in $f\{0, n'\}$ and let $\phi : f\{0, 2d\} \to \{c_1, c_2, \cdots, c_m\}$ be a one-to-one onto mapping. Let $g = (g(0), g(1), \cdots, g(n' + 2d))$ be the sequence defined as $g(i) = f(i)$ for $i \leq n' - 1$ and $g(i) = \phi(f(i - n'))$ for $n' \leq i \leq n' + 2d$. It is straightforward to verify that g is a partial clique covering sequence of G and $g[0, 2d] \equiv g[n', n' + 2d]$. Moreover, $AS_{g[0,n'-1]} = S_{f[0,n'-1]} > S(D) - \epsilon$. $\square$

Let $g = (g(0), g(1), \cdots, g(q + 2d))$ be a partial clique covering sequence of G of minimum length such that $g[0, 2d] \equiv g[q, q + 2d]$ and $AS_{g[0,q-1]} > S(D) - \epsilon$. By the choice of ϵ, we know that $q > \ell$.

It is obvious that each partial clique covering sequence of G with length $2d + 1$ is isomorphic to a partial clique covering sequence with all entries in the set $\{1, 2, \cdots, 2d+1\}$. Hence there are at most $(2d+1)^{2d+1}$ non-isomorphic partial clique covering sequences of G of length $2d + 1$.

Consider the set $\{g[i, i + 2d] : i = 2d + 1, 2d + 2, \cdots, q - 2d\}$ of partial clique covering sequences of G. Since there are $q-4d \geq \ell-4d > (2d+1)^{2d+1}$ such sequences, we conclude that there exists $i < j$ such that $g[i, i+2d] \equiv g[j, j+2d]$. Thus $g[i, j-1]$ is a complete segment of g. Let g' be obtained from g by cutting off the complete segment $g[i, j-1]$. It follows from Lemma 2 and Corollary 3 that g' is a partial clique covering sequence of G and $AS_{g'[0,q-(j-i)-1]} \geq AS_{g[0,q-1]} > S(D) - \epsilon$, contrary to the choice of g. Therefore, we conclude that there exists a partial clique covering sequence $g = (g(0), g(1), \cdots, g(q + 2d))$ with $g[0, 2d] \equiv g[q, q + 2d]$ and $q \leq (2d + 1)^{2d+2}$ such that $S(D) = AS_g$. Q.E.D.

Corollary 6 *For any finite D-set, the asymptotic clique covering ratio $S(D)$ is a rational number.*

Theorem 5 also shows that the limit $\lim_{n\to\infty} \frac{n}{\chi(\overline{G}(n,D))}$ exists, and hence $S(D) = \lim_{n\to\infty} \frac{n}{\chi(\overline{G}(n,D))}$.

4 Distance graphs with $S(D) = \omega(G(Z,D))$

As observed in Section 1, we have $S(D) \leq \omega(G(Z,D))$ for all distance sets D. There are distance sets D for which the strict inequality holds, $S(D) < \omega(G(Z,D))$. For example, if $D = \{2,3,5,8\}$, then $\omega(G(Z,D)) = 4$. The only type of K_4 is of the form $\{i, i+3, i+5, i+8\}$. It is easy to see that the vertex set of $G(Z,D)$ can not be partitioned into this only type of K_4's. Therefore $S(D) < 4$. Indeed, for any two distinct odd integers x, y, if $D = \{x, y, y-x, y+x\}$, then $\omega(G(Z,D)) = 4$ and $S(D) < 4$. In this section, we investigate the following question:

Question 1 *For which D the equality $S(D) = \omega(G(Z,D))$ holds ?*

In other words, we consider for what distance graphs whose vertices can be partitioned into cliques of maximum size ?

We first define the fractional chromatic number and the circular chromatic number of a graph which are needed in the discussion. The *fractional chromatic number* $\chi_f(G)$ of a graph G is the minimum total weight that can be assigned to the independent sets of G so that for each vertex x the total weight of those independent sets containing x is at least 1. A (k,d)*-coloring* of a graph G is a function that assigns to each vertex a color from the set $\{0, 1, \cdots, k-1\}$ such that $d \leq |c(x) - c(y)| \leq k - d$ for every edge xy of G. The *circular chromatic number* $\chi_c(G)$ of G is the minimum ratio k/d if G has a (k,d)-coloring. The following are known [33]:

$$\omega(G) \leq \chi_f(G) \leq \chi_c(G) \leq \chi(G) \text{ and } \lceil \chi_c(G) \rceil = \chi(G) \qquad (*).$$

Now we relate Question 1 to a class of distance graphs which arose from the study in T-colorings. A T-set has property (**) if the following is true:

$$\mathrm{sp_T(G)} = \mathrm{sp_T(K_{\chi(G)})}, \text{ for all graphs } G \qquad (**)$$

The problem about which T-sets have property (**) was studied by several authors [5, 27, 21, 22, 23]. Our next result shows that this problem is related to Question 1.

Theorem 7 *If T has the property (**) and let $D = T - \{0\}$, then $S(D) = \omega(G(Z,D)) = \chi_f(G(Z,D))$.*

Proof. It was proved in [21] that T has the property (**) if and only

if $\chi(\overline{G}(n,D)) = \omega(\overline{G}(n,D))$ for all n, where $D = T - \{0\}$. It was proved in [3] that $\chi_f(G(Z,D)) = \lim\limits_{n\to\infty} \frac{n}{\omega(\overline{G}(n,D))}$. Therefore we have

$$\begin{aligned}
S(D) &= \lim_{n\to\infty} \frac{n}{\chi(\overline{G}(n,D))} \\
&\leq \omega(G(Z,D)) \\
&\leq \chi_f(G(Z,D)) \\
&= \lim_{n\to\infty} \frac{n}{\omega(\overline{G}(n,D))} \\
&= \lim_{n\to\infty} \frac{n}{\chi(\overline{G}(n,D))} \\
&= S(D).
\end{aligned}$$

Therefore $S(D) = \omega(G(Z,D)) = \chi_f(G(Z,D))$. Q.E.D.

The converse of Theorem 7 is not always true. There are sets D such that $S(D) = \omega(G(Z,D))$ but $\mathrm{sp_T(G)} \neq \mathrm{sp_T(K_{\chi(G)})}$ for some graph G, where $T = D \cup \{0\}$. As an example, let $D = \{a, a+1, \cdots, b\}$. We prove in the next result that for such distance sets D, $S(D) = \omega(G(Z, D))$ always holds. However, it was proved in [22] that if $T = \{0, a, a + 1, \cdots, b\}$, then T has the property (**) if and only if b is a multiple of a.

Theorem 8 *If* $D = \{a, a + 1, \cdots, b\}$, *then* $S(D) = \omega(G(Z, D)) = \lfloor \frac{b}{a} \rfloor + 1$.

Proof. It was proved in [2] that $\chi_c(G(Z, D)) = 1 + \frac{b}{a}$. Therefore $\omega(G(Z, D)) \leq \lfloor \chi_c(G(Z, D)) \rfloor = \lfloor \frac{b}{a} \rfloor + 1$. As $\{0, a, 2a, \cdots, \lfloor \frac{b}{a} \rfloor a\}$ induces a clique of $G(Z, D)$, we conclude that $\omega(G(Z, D) = \lfloor \frac{b}{a} \rfloor + 1$.

Let $m = \lfloor \frac{b}{a} \rfloor + 1$. To complete the proof, it suffices to show that the vertices of $G(ma, D)$ can be covered by a cliques. Define $A_i = \{i + ja : j = 0, 1, 2, \cdots, m - 1\}$, $i = 0, 1, 2, \cdots, a - 1$. Then each A_i is a clique and the = vertices of $G(ma, D)$ is the disjoint union of the A_i's, $0 \leq i \leq a - 1$. Q.E.D.

The coloring of distance graphs has been studied extensively [2, 3, 4, 6, 7, 10, 11, 12, 13, 19, 8, 9, 24, 25, 30, 29, 32, 34, 35]. It seems unlikely that some general coloring method can find the chromatic number of all distance graphs. However, there is a very simple general coloring method that works for many distance graphs: the regular coloring method. This method was used to determine not only the chromatic number but also the circular chromatic number of many distance graphs [2, 3, 6, 14, 20, 35]. The essence of the regular coloring method is revealed in the proof of Theorem 9 below, which was proved in [31]. In order to explain this coloring method, we include a proof here.

For any real number x, let $||x||$ denote the distance from x to the nearest integer. Suppose D is a finite set of positive integers and r is a real number, let $||rD|| = \min\{||rx|| : x \in D\}$. For a finite set D of positive integers, we define the function $\kappa(D)$ as

$$\kappa(D) = \sup_{r \in R} ||rD||.$$

Theorem 9 *For a finite set D of positive integers,*

$$\chi_c(G(Z,D)) \leq \frac{1}{\kappa(D)}.$$

Proof. It is not difficult to see that when D is a finite set of positive integers, then $\kappa(D)$ is rational. Suppose $\kappa(D) = ||\frac{q}{p}D|| = \frac{d}{k}$. Let $r = \frac{p}{qk}$. We partition the real line R into half open intervals $I_i = [ir, (i+1)r)$. Then color the interval I_i with color $i \pmod k$. Intuitively, we make a "brush" of width r and paint the line R with this brush and with k colors each used once in every k consecutive brushings. This coloring of a distance graph is called a *regular k-coloring* with parameter r.

Now we show that the restriction of this coloring to Z is a (k,d)-coloring of $G(Z,D)$. By definition, the color $c(i)$ of vertex i is given by the formula

$$c(i) = \lfloor \frac{qki}{p} \rfloor \pmod k.$$

Then for any edge ij of $G(Z,D)$, $|i-j| \in D$. Hence

$$\begin{aligned} |c(i)-c(j)| &= |\lfloor \frac{qki}{p} \rfloor \pmod k - \lfloor \frac{qkj}{p} \rfloor \pmod k| \\ &> \lfloor \frac{qk|i-j|}{p} \rfloor \pmod k - 1 \\ &\geq d-1. \end{aligned}$$

Therefore $|c(i)-c(j)| \geq d$. Similarly we can prove that $|c(i)-c(j)| \leq k-d$. Therefore c is a (k,d)-coloring of $G(Z,D)$ and hence $\chi_c(G(Z,D)) \leq k/d$. Q.E.D.

To determine the circular chromatic number of $G(Z,D)$ by using the regular coloring method, one usually proves that $\frac{1}{\kappa(D)}$ is a lower bound for the fractional chromatic number, i.e., $\chi_f(G(Z,D)) \geq \frac{1}{\kappa(D)}$. Combining this with (*), we obtain $\chi_f(G(Z,D)) = \chi_c(G(Z,D)) = \frac{1}{\kappa(D)}$.

Our next result shows that $S(D) = \omega(G(Z,D))$, if $\frac{1}{\kappa(D)}$ is equal to the clique number of $G(Z,D)$.

Theorem 10 *If* $\omega(G(Z,D)) = \frac{1}{\kappa(D)}$, *then* $S(D) = \omega(G(Z,D)) = \chi_f(G(Z,D)) = \chi_c(G(Z,D)) = \chi(G(Z,D)) = \frac{1}{\kappa(D)}$

Proof. Assume $\omega(G(Z,D)) = \frac{1}{\kappa(D)}$. It follows from (*) and Theorem 9 that $\chi_f(G(Z,D)) = \chi_c(G(Z,D)) = \chi(G(Z,= D)) = \frac{1}{\kappa(D)}$. We shall show that $S(D) = \omega(G(Z,D))$.

Let $m = \omega(G(Z,D))$. Suppose $\kappa(D) = ||\frac{q}{mp}D|| = \frac{1}{m}$, where p,q are positive integers. We choose p and q so that $mp \geq \max\{x : x \in D\}$ (thus, mp and q are not necessarily coprime). By the proof of Theorem 9, the mapping f defined by $f(i) = \lfloor \frac{qi}{p} \rfloor \pmod m$ is an m-coloring of $G(Z,D)$. Similarly, the mapping ψ defined as

$$\psi(i) = qi \pmod{mp}$$

is an (mp,p)-coloring of $G(Z,D)$.

Let $G_{mp}(D)$ be the circulant graph with vertices $0,1,\cdots,mp-1$ and uv is an edge in $G_{mp}(D)$ if and only if $|u-v| \in D$ or $mp - |u-v| \in D$. The restriction of ψ to the vertices of $G_{mp}(D)$ is an (mp,p)-coloring of $G_{mp}(D)$. With an abuse of the notation, we regard ψ as a homomorphism from $G_{mp}(D)$ to G^p_{mp}, which has vertices $0,1,\cdots,mp-1$ and ij is an edge if and only if $p \leq |i-j| \leq (m-1)p$.

Claim. $G_{mp}(D)$ can be covered by p cliques of size m.

Proof. By our assumption, $\omega(G(Z,D)) = m$. As $G(Z,D)$ is vertex transitive and by the choice of p, $G(Z,D)$ has a clique $Y_0 = \{y_0,y_1,y_2,\cdots,y_{m-1}\}$ such that $0 \leq y_0 < y_1 < \cdots < y_{m-1} \leq mp-1$. By the definition of $G_{mp}(D)$, Y_0 is also a clique of $G_{mp}(D)$, hence $\psi(Y_0)$ is a clique of G^p_{mp}. It is easy to verify that every clique of G^p_{mp} of size m is of the form $\{i, i+p, i+2p, \cdots, i+(m-1)p\}$ for some i. Therefore without loss of generality we may assume that $y_j \in \psi^{-1}(jp)$ for $j = 0,1,\cdots,m-1$.

If $gcd(mp,q) = 1$, then $|\psi^{-1}(i)| = 1$ for all i. Then $G_{mk}(D)$ is covered by the p maximum cliques: $\{\psi^{-1}(i), \psi^{-1}(i+p) \cdots, \psi^{-1}(i+(m-1)p)\}$, $i = 0,1,2,\cdots,p-1$. If $gcd(mp,q) = d$, then $|\psi^{-1}(i)| = d$ if $d|i$; and $|\psi^{-1}(i)| = 0$ otherwise. To complete the proof of the claim, it suffices to show that if $d|j$, then $\psi^{-1}(j) \cup \psi^{-1}(j+p) \cup \cdots \cup = \psi^{-1}(j+(m-1)p)$ is a disjoint union of d maximum cliques in $G_{mp}(D)$.

It follows from the definition that $\psi(i) = j$ if and only if $\psi(i + \frac{mp}{d}) = j$, where the summation is carried out modulo mp. Therefore $\psi^{-1}(x) = \{x_0, x_0 + \frac{mp}{d}, x_0 + \frac{2mp}{d}, \cdots, x_0 + \frac{(d-1)mp}{d} \pmod{\frac{mp}{d}}\}$, where x_0 is any vertex of $\psi^{-1}(x)$.

Now we can construct d disjoint cliques Y_i in $G_{mp}(D)$ by letting $Y_i = Y_0 + \frac{imp}{d}$ for $0 \leq i \leq d-1$, where the summation is carried out modulo mp. Similarly, by shifting these cliques, one can show that $\psi^{-1}(j), \psi^{-1}(j+p), \cdots, \psi^{-1}(j+(m-1)p)$ is a disjoint union of d maximum cliques in $G_{mp}(D)$. The images of these cliques under the homomorphism ψ cover all vertices of G^p_{mp}, and all these cliques cover all vertices of $G_{mp}(D)$. To be precise, the vertices of $G_{mp}(D)$ are covered by p cliques $Y_{i,j} = Y_0 + \frac{imp}{d} + j$ for $0 \leq i \leq d-1$ and $0 \leq j \leq p/d - 1$, where the additions are carried out modulo mp. This completes the proof of the Claim. $\square$

Since Y_0 is a clique in $G(Z, D)$, $Y_{i,j} = Y_0 + x$ (where the addition is the ordinary addition, i.e., does not take modulo) are cliques of G. Therefore the vertices of $G(Z, D)$ are covered by the cliques $Y_{i,j} = Y_0 + \frac{imp}{d} + j$ for $i = \cdots, -2, -1, 0, 1, 2, \cdots$ and $0 \leq j \leq p/d - 1$. Q.E.D.

There are many D-sets for which $\omega(G(Z, = D)) = \frac{1}{\kappa(D)}$, such as $D = \{a, a+1, \cdots, ka\}$, and $D = \{a, b, a+b\}$, $\gcd(a, b) = 1$ and $a \equiv b \pmod 3$ [2, 31], etc. There are also many distance sets D for which the answer to Question 1 is positive, but $\omega(G(Z, D)) \neq \frac{1}{\kappa(D)}$. The class of distance graphs we consider below provides such examples.

Given positive integers m, k and s with $m > ks$, let $D_{m,k,s} = \{0, 1, 2, \cdots, m\} - \{k, 2k, 3k, \cdots, sk\}$. The distance graphs with D-sets $D = D_{m,k,s}$ have been studied by several authors [10, 19, 24, 3, 25, 7, 18, 34]. Erdős, Eggleton and Skelton initiated the study of this family of distance graphs for the case that $s = 1 =$ [10]. Chang, Liu and Zhu [3] gave the fractional chromatic number of $G(Z, D_{m,k,1})$, and completely determined the value of $\chi(G(Z, D_{m,k,1}))$ for any m and k.

The distance graphs with distance sets $D = D_{m,k,s}$ for other values of s were first studied by Liu and Zhu [25] in which the authors showed the results of fractional chromatic number: $\chi_f(G(Z, D_{m,k,s})) = k$ if $m < (s+1)k$; and $\chi_f(G(Z, D_{m,k,s}) = (m+sk+1)/(s+1)$ if $m \geq (s+1)k$, and determined $\chi(G(Z, D_{m,k,s}))$ for the cases $s = 2, 3$ and some others. The exact values of $\chi(G(Z, D_{m,k,s}))$ for all m, k, and s were completely solved by Huang and Chang [18]. The circular chromatic number of $G(Z, D_{m,k,s})$ was investigated in [3, 18, 34], and the complete solution was recently obtained by Zhu [34].

Suppose $D = D_{m,k,s}$. Let $m' = m+sk+1$ and let $d = \gcd(m', k) = \gcd(m+1, k)$. The following result was proved in [34]:

Theorem 11

$$\chi_c(G) = \begin{cases} m'/(s+1), & \textit{if } d = 1 \quad \textit{or} \quad d(s+1) \mid m', \\ (m'+1)/(s+1), & \textit{otherwise.} \end{cases}$$

A consequence of this result is that for some distance sets $D = D_{m,k,s}$ (i.e., those $D_{m,k,s}$ for which $d \neq 1$ and $d(s+1) \not| m'$), $\chi_f(G) \neq \chi_c(G)$. Therefore for these distance graphs, $\chi(G) \neq \omega(G)$. However, our next result shows that for all distance sets $D = D_{m,k,s}$, $S(D) = \omega(G(Z, D))$.

Theorem 12 *Suppose $D = D_{m,k,s} = \{0, 1, 2, \cdots, m\} - \{k, 2k, 3k, \cdots, sk\}$ where $m = (s+1)kq + r$ for some nonnegative integers q and r with $0 \leq r \leq (s+1)k - 1$. Then*

$$S(D) = \omega(G(Z, D)) = \begin{cases} qk + r + 1, & \textit{if } 0 \leq r \leq k-1; \\ (q+1)k, & \textit{if } k \leq r \leq (s+1)k - 1. \end{cases}$$

Proof. We first prove the second equality. By definition of $D = D_{m,k,s}$, it is easy to verify that the maximum size of a clique in any set of consecutive $(s+1)k$ vertices in $G(Z, D)$ is k, i.e. $\omega(G((s+1)k, D)) = k$. Therefore, to show the second equality it suffices to find cliques in $G(Z, D)$ with the desired cardinalities. If $0 \leq r \leq k-1$, the set of vertices $(\cup_{i=0}^{q-1}\{i(s+1)k, i(s+1)k+1, \cdots, i(s+1)k+k-1\}) \cup \{q(s+1)k, q(s+$

$1)k+1, \cdots, q(s+1)k+r\}$ forms a clique in G; if $k \le r \le (s+1)k-1$, then the set of vertices $\cup_{i=0}^{q}\{i(s+1)k, i(s+1)k+1, \cdots, = i(s+1)k+k-1\}$ is a clique in G. These prove the second equality.

Now we show the first equality. Suppose $k \le r \le (s+1)k-1$, it suffices to prove that the vertices of $G((q+1)(s+1)k, D)$ can be covered by $(s+1)$ cliques. For $j = 0, 1, 2, \cdots, s$, let $A_j = \cup_{i=0}^{q}\{i(s+1)k+jk, i(s+1)k+jk+1, \cdots, = i(s+1)k+jk+k-1\}$. It is easy to see that each A_j is a clique and the vertices of $G((q+1)(s+1)k, D)$ is the disjoint union of the A_j's, $j = 0, 1, \cdots, s$.

If $0 \le r \le k-1$, it suffices to prove that the vertices of $G((s+1)(qk+r+1), D)$ can be covered by $(s+1)$ cliques. Consider the case $q = 1$. We partition the set of vertices $\{0, 1, \cdots, (s+1)(k+r+1)-1\}$ into $(3s+3)$ blocks (i.e. intervals of consecutive vertices) by the following three steps:

(1) each of the first $(s+1)$ blocks, A_0, A_1, A_2, $\cdots$, A_s, has length $r+1$;

(2) each of the next $(s+1)$ blocks, B_0, B_1, B_2, $\cdots$, B_s, has length $k-r-1$;

(3) each of the last $(s+1)$ blocks, C_0, C_1, C_2, $\cdots$, C_s, has length $r+1$.

Let $W_i = A_i \cup B_{s-i} \cup C_i$ for $i = 0, 1, 2, \cdots, s$.

Then each W_i is a clique in G. (For instance, $W_0 = A_0 \cup B_s \cup C_0$, where $A_0 = \{0, 1, \cdots, r\}$, $B_s = \{(s+1)(r+1)+s(k-r-1), (s+1)(r+1)+s(k-r-1)+1, \cdots, (s+1)k-1\}$, and $C_0 = \{(s+1)k, (s+1)k+1, \cdots, (s+1)k+r\}$.) This implies that the vertices of $G((s+1)(k+r+1), D)$ can be covered by $s+1$ cliques.

For the case that $q \ge 2$, one can extend the method above by first repeating (1) and (2) together for q times and then adjoining (3). To be precise, we partition the vertices of $G((s+1)(qk+r+1), D)$ into blocks $A_{i,j}$, $B_{i,j}$ and C_i $(i, j = 0, 1, 2 \cdots, s)$

in the following order:

$$\begin{array}{llll}
A_{0,0} \to & A_{0,1} \to & \cdots & \to A_{0,s} \\
\to B_{0,0} \to & B_{0,1} \to & \cdots & \to B_{0,s} \\
\to A_{1,0} \to & A_{1,1} \to & \cdots & \to A_{1,s} \\
\to B_{1,0} \to & B_{1,1} \to & \cdots & \to B_{1,s} \\
\cdots & & & \\
\to A_{s,0} \to & A_{s,1} \to & \cdots & \to A_{s,s} \\
\to B_{s,0} \to & B_{s,1} \to & \cdots & \to B_{s,s} \\
\to C_0 \to & C_1 \to & \cdots & \to C_s
\end{array}$$

where each A or C block has $(r+1)$ vertices and each B block has $(k-r-1)$ vertices. For $0 \leq i \leq s$, let $W_i = \cup_{j=0}^{s}(A_{i,j} \cup B_{i,s-j}) \cup C_i$. Then each W_i is a clique in G and the vertices of $G((s+1)(qk+r+1), D)$ are covered by W_i, $0 \leq i \leq s$. Q.E.D.

References

[1] D. G. Cantor and B. Gordon, *Sequences of integers with missing differences*, J. Combinatorial Theory, Series A, 14 (1973), 281-287.

[2] G. J. Chang, L. Huang and X. Zhu, *Circular chromatic numbers and fractional chromatic numbers of distance graphs*, Europ. J. of Comb., 19 (1998), 423-431.

[3] G. J. Chang, D. D.-F. Liu and X. Zhu, *Distance graphs and T-coloring*, J. of Comb. Theory, Series B, 75 (1999), 259 - 269.

[4] J. J. Chen, G. J. Chang and K. C. Huang, *Integral distance graphs*, J. Graph Theory, 25 (1997), 287-294.

[5] M. B. Cozzens and F. S. Roberts, *T-colorings of graphs and the channel assignment problem*, Congr. Numer., 35 (1982), 191-208.

[6] W. Deuber and X. Zhu, *The chromatic number of distance graphs*, Disc. Math., 165/166 (1997), 195-204.

[7] W. Deuber and X. Zhu, *Chromatic numbers of distance graphs with distance sets missing multiples*, manuscript, 1997.

[8] R. B. Eggleton, *New results on 3-chromatic prime distance graphs*, Ars Comb., 26 (1988), 153-180.

[9] R. B. Eggleton, *Distance graphs and their chromatic numbers*, Workshop Notes of the Conference of Graph Theory and Combinatorics, Manila (1991).

[10] R. B. Eggleton, P. Erdős and D. K. Skilton, *Coloring the real line*, J. Comb. Theory, Series B, 39 (1985), 86-100.

[11] R. B. Eggleton, P. Erdős and D. K. Skilton, *Research problem 77*, Disc. Math., 58 (1986), 323.

[12] R. B. Eggleton, P. Erdős and D. K. Skilton, *Update information on research problem 77*, Disc. Math., 69 (1988), 105.

[13] R. B. Eggleton, P. Erdős and D. K. Skilton, *Coloring prime distance graphs*, Graphs and Comb., 32 (1990), 17-32.

[14] G. Gao and X. Zhu, *Star extremal graphs and the lexicographic product*, Disc. Math., 152 (1996), 147-156.

[15] J. R. Griggs and D. D.-F. Liu, *The channel assignment problem for mutually adjacent sites*, J. Comb. Theory, Series A, 68 (1994), 169-183.

[16] W. K. Hale, *Frequency assignment: theory and applications*, Proc. IEEE, 68 (1980), 1497-1514.

[17] N. M. Haralambis, *Sets of integers with missing differences*, J. Comb. Theory, Series A, 23 (1977), 22-23.

[18] L. Huang and G. J. Chang, *Circular chromatic number of distance graphs with distance set missing multiples*, Europ. J. of Comb., 20 (1999), 1-8.

[19] A. Kemnitz and H. Kolberg, *Coloring of integer distance graphs*, Tech. Report 96/07, Institute für Mathematik, Technische Universität Braunschweig (1996).

[20] K.-W. Lih, D. D.-F. Liu and X. Zhu, *Star extremal circulant graphs*, SIAM J. Disc. Math., 12 (1999), 491-499.

[21] D. D.-F. Liu, *T-colorings of graphs*, Disc. Math., 101 (1992), 203-212.

[22] D. D.-F. Liu, *On a conjecture of T-colorings*, Congr. Numer., 103 (1994), 27-31.

[23] D. D.-F. Liu, *T-graphs and the channel assignment problem*, Disc. Math. 161 (1996) 197-205.

[24] D. D.-F. Liu, *T-coloring and chromatic number of distance graphs*, Ars. Comb., ***.

[25] D. D.-F. Liu and X. Zhu, *Distance graphs with missing multiples in the distance sets*, Jour. of Graph Theory, 30 (1999), 245 - 259.

[26] J. H. Rabinowitz and V. K. Proulx, *An asymptotic approach to the channel assignment problem*, SIAM J. Alg. Disc. Math., 6 (1985), 507-518.

[27] A. Raychaudhuri, *Further results on T-coloring and frequency assignment problem*, SIAM J. Disc. Math., 7 (1994), 605-613.

[28] B. A. Tesman, *T-colorings, List T-colorings and set T-Colorings of Graphs*, Ph.D. Dissertation, Dept. Math., Rutgers University (1989).

[29] M. Voigt and H. Walther, *Chromatic number of prime distance graphs*, Disc. Appl. Math., 51 (1994), 197-209.

[30] M. Voigt, *Coloring of distance graphs*, Ars Comb., to appear. ??? (it appeared)

[31] X. Zhu, *Distance graphs on the real line*, manuscript (1996).

[32] X. Zhu, *Pattern-periodic coloring of distance graphs*, J. Comb. Theory B, 73 (1998), 195-206.

[33] X. Zhu, *Circular chromatic number: a survey*, Disc. Math., 229 (2001), 371-410.

[34] X. Zhu, *Circular chromatic numbers of a class of distance graphs*, manuscript, 1998.

[35] X. Zhu, *Circular chromatic number of distance graphs with distance set of cardinality* 3, Journal of Graph Theory, to appear.