HOW TO WIN IN SLOW EXACT k-NIM

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Slow Exact k-NIM SN(n, k)

- Play on n stacks of tokens
- Move consists of
  - Picking exactly k of the stacks
  - Removing one token from each of the selected stacks
- Last person to make a move wins

Known Results

Gurvich et al. [2020] Slow k-Nim.

Chickin et al. [2021] More about Slow Exact k-Nim.

$n \backslash k$	1	2	3	4	5	6	…
1	SN(1,1)						
2	SN(2,1)	SN(2,2)					
3	SN(3,1)	SN(3,2)	SN(3,3)				
4	SN(4,1)	SN(4,2)	?	SN(4,4)			
5	SN(5,1)	SN(5,2)	?	?	SN(5,5)		
6	SN(6,1)	SN(6,2)	?	?	?	SN(6,6)	
7	SN(7,1)	?	?	?	?	?	
⋮	⋮	⋮	⋮	⋮	⋮	⋱	⋱

- Two infinite families of games: **SN(n, 1)** and **SN(n, n)** which are deterministic.
- P-positions are
  - **SN(n, 1):** **sum(p)** is **even**.
  - **SN(n, n):** **min(p)** is **even**.

Our Results

$n \backslash k$	1	2	3	4	5	6	...
1	SN(1,1)						
2	SN(2,1)	SN(2,2)					
3	SN(3,1)	SN(3,2)	SN(3,3)				
4	SN(4,1)	SN(4,2)	SN(4,3)	SN(4,4)			
5	SN(5,1)	SN(5,2)	?	SN(5,4)	SN(5,5)		
6	SN(6,1)	SN(6,2)	?	?	SN(6,5)	SN(6,6)	
7	SN(7,1)	?	?	?	?	SN(7,6)	
⋮	⋮	⋮	⋮	⋮	⋮	⋱	⋱

P-positions of a non-trivial infinite family, **SN$(n, n-1)$**, where play is on **all but one stack**

Results for P-Positions of Slow Exact k-NIM, $k = n - 1$

Characterization of P-positions:

- $\boldsymbol{s}$ = sum($\boldsymbol{p}$) mod $\boldsymbol{d}$ where $\boldsymbol{d} = \boldsymbol{k}$ when n **is even** and $\boldsymbol{d} = \boldsymbol{2k}$ when n **is odd**
- $\boldsymbol{o}$ = # of stacks with odd stack heights

$n = 10, k = 9, s = \mathrm{sum}(\boldsymbol{p}) \bmod 9$

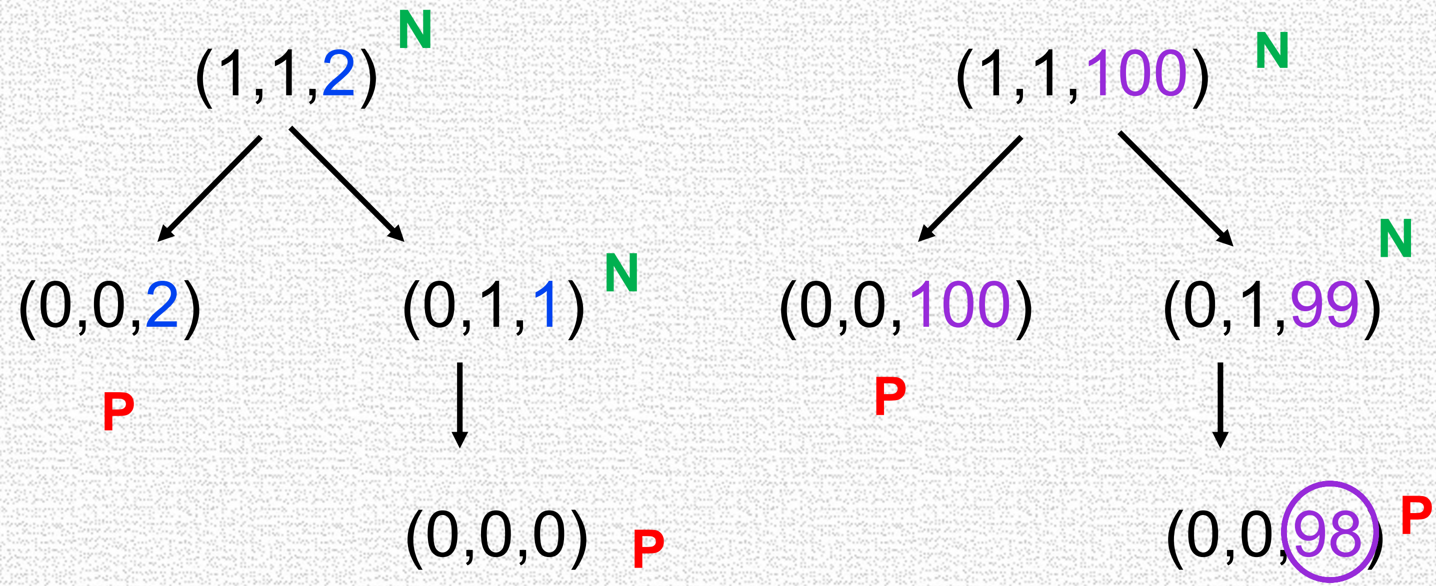
s\o	0	1	2	3	4	5	6	7	8	9	10
0											
1											
2											
3											
4											
5											
6											
7											
8											

$n = 9, k = 8, s = \mathrm{sum}(\boldsymbol{p}) \bmod 16$

s \o	0	1	2	3	4	5	6	7	8	9
0										
1										
2										
3										
4										
5										
6										
7										
8										
9										
10										
11										
12										
13										
14										
15										

These results are for **REDUCED** positions.

Reduced Positions – Motivation – SN(3,2)



Game trees are isomorphic – same outcome

p = (1,1,2) is the **reduced** position for **p** = (1,1,2+m) with $m \geq 0$

Reduced Positions - Definition

Definition: A position is **reduced** if, for each stack, there exists a sequence of legal moves that deplete the stack.

Difficult to check, not useful for proofs

Theorem

A position is reduced **if and only if** the **NIRB (No stack Is Really Big) condition**

$$\max(\boldsymbol{p}) \leq \frac{\text{sum}(\boldsymbol{p})}{k}$$

is satisfied.

Outline of Proof of our Results

- Reduce initial position using the NIRB condition repeatedly
- For each position, a move is either to a reduced position or a position that needs to be reduced.
- Characterize when reduction is needed and what reduced position looks like

- Then show that
  - from each P-position, all moves lead to N-positions
  - from each N-position, there is at least one move to a P-position

s\o	0	1	2	3	4	5	6	7	8	9	10
0											
1											
2											
3											
4											
5											
6											
7											
8											

Reduction Criterion for even n

$\boldsymbol{p} = (p_1, p_2, \dots, p_n)$ with $p_1 \leq p_2 \leq \dots \leq p_n$

Lemma: When reduction is needed from a position characterized by (s, o) and $\boldsymbol{p}$ has $\alpha \geq 1$ maximal stacks, then the reduced position is given by

$$r(\boldsymbol{p}') = \begin{cases} (p_1 - 1, p_2 - 1, \dots, p_n - 1) & \text{if } s > 0 \\ (p_1 - 1, p_2 - 1, \dots, p_{n-\alpha} - 1, p_n - 2, \dots, p_n - 2) & \text{if } s = 0 \end{cases}$$

- If $s > 0$, then $(s', o') = (s - 1, n - o)$;
- If $s = 0$, then $\alpha \leq n - 2$ and

$$s' = n - 2 - \alpha \text{ and } o' = \begin{cases} n - o - \alpha & \text{if } p_n \text{ is even} \\ n - o + \alpha & \text{if } p_n \text{ is odd} \end{cases}.$$

Illustration for n even – P-Position leads to N-Position

$\boldsymbol{s}$ = sum($\boldsymbol{p}$) mod $\boldsymbol{k}$
$\boldsymbol{o}$ = # of odd stacks

3 cases:

- Move to position that is reduced
- Move to a position that needs reduction; $s > 0$ and $s = 0$

<u>Case 1</u>: No reduction → remove exactly k tokens:

- $s' = s$ → same row;
- $o' \in \{n - o - 1, n - o + 1\}$ depending on parity of un-played stack

→ reflect cell across column ℓ and then go either left or right

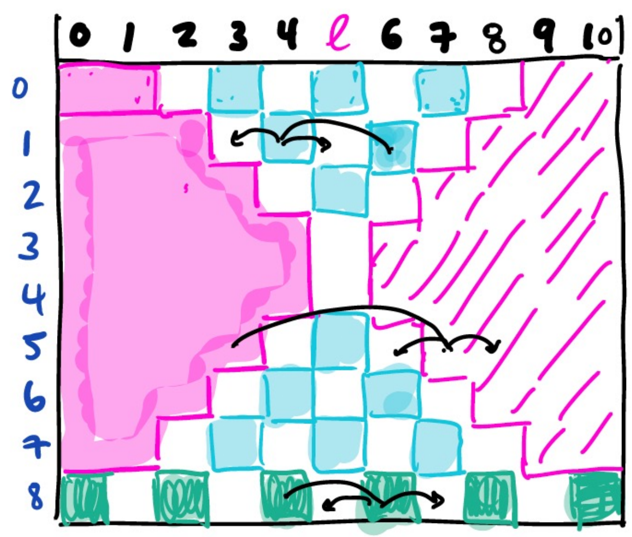
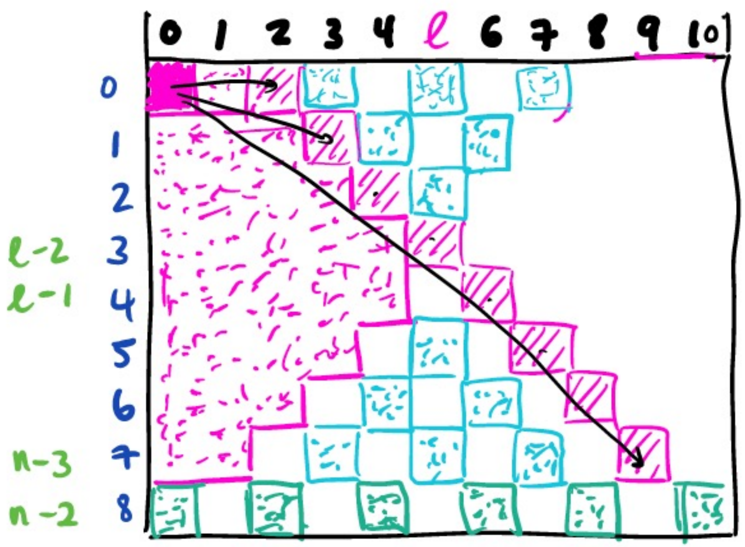


Illustration for n even – P-Position leads to N-Position

Case 3: Move is to a position that needs reduction when $s = 0$

- Take a token from all stacks, and an additional token from the maximal stacks
- $s' = n - 2 - \alpha$ and $o' = \begin{cases} n - o - \alpha & \text{if } p_n \text{ is even} \\ n - o + \alpha & \text{if } p_n \text{ is odd} \end{cases}$ with $1 \leq \alpha \leq n - 2$



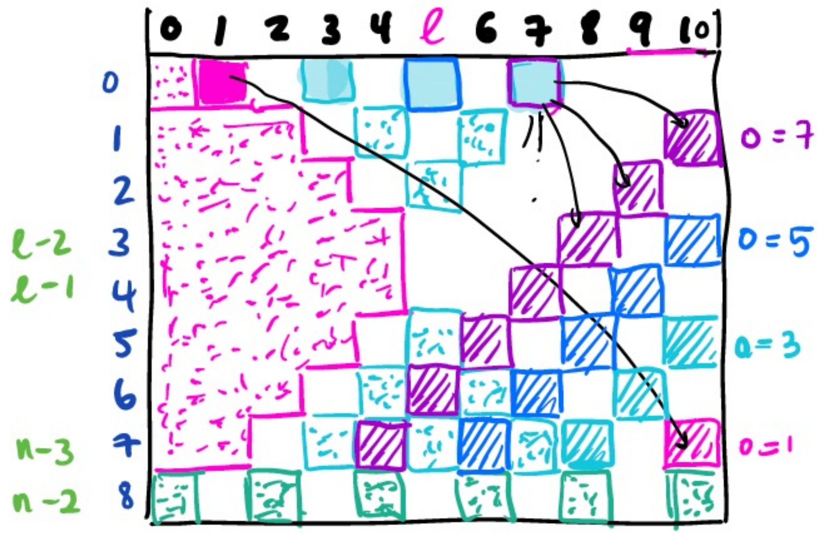
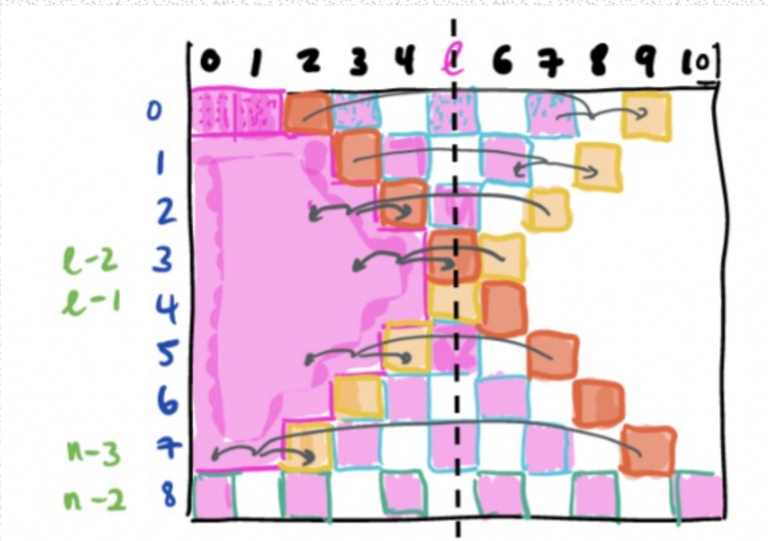


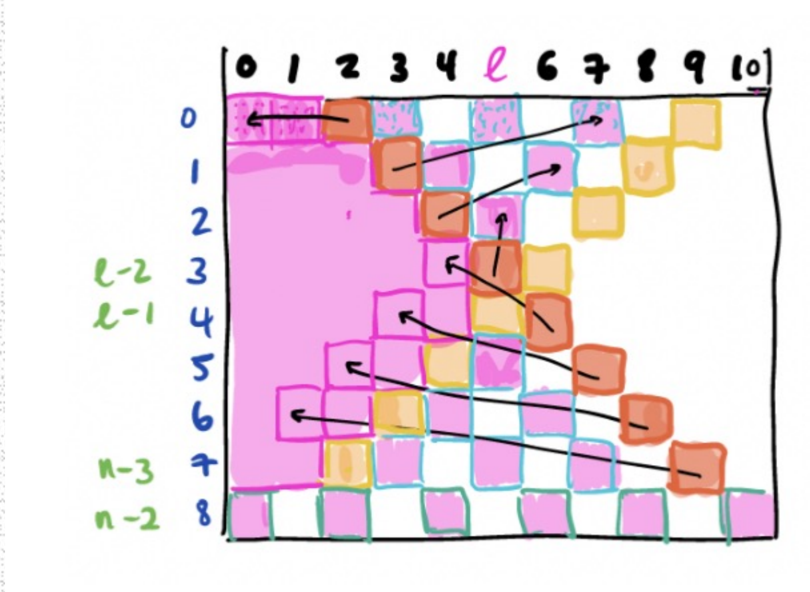
Illustration for n even: N-position to P-position

- No reduction – same row, reflection across midline, then to left or right depending on parity of max
- Yellow and orange squares are the only ones with potential trouble



Yellow squares can be shown to have non-reduction move available

For orange squares need to check on reduction move, and it turns out this one is available.



Ongoing and Future Work

Generalization to Slow SetNIM SN$(n, \boldsymbol{A})$, where the set $\boldsymbol{A}$ indicates the possible numbers of stacks one can play on

✓Develop a NIRB condition for the game with set $\boldsymbol{A}$

$$\max(\boldsymbol{p}) \leq \frac{\mathrm{sum}(\boldsymbol{p})}{k}, \qquad \text{where } k = \min(\boldsymbol{A})$$

- Develop characterization when reduction is needed
  - depends on the individual game
- Determine what reduced position looks like
  - depends on the individual game

✓Analyze some games

- we have the result for $A = \{n-1, n\}$

Results for P-Positions of Slow SetNIM SN$(n, \{\boldsymbol{n-1}, \boldsymbol{n}\})$

$n = 10, k = 9, s = \text{sum}(\boldsymbol{p}) \bmod 9$

s\o	0	1	2	3	4	5	6	7	8	9	10
0											
1											
2											
3											
4											
5											
6											
7											
8											X

o = # of stacks with odd stack heights

- Looks very much like SN$(n, n-1)$
- Only difference: P-positions with $o = n$ are removed

$n = 9, k = 8, s = \text{sum}(\boldsymbol{p}) \bmod 16$

s \o	0	1	2	3	4	5	6	7	8	9
0										
1										
2										
3										
4										
5										
6										
7										X
8										
9										
10										
11										
12										
13										
14										
15										

THANK YOU!

Any questions?

You can reach me at sheubac@calstatela.edu

References

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Image citation

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