Circular consecutive choosability of k-choosable graphs

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Abstract

Let $S(r)$ denote a circle of circumference r. The circular consecutive choosability $ch_{cc}(G)$ of a graph G is the least real number t such that for any $r \geqslant \chi_c(G)$, if each vertex v is assigned a closed interval $L(v)$ of length t on $S(r)$, then there is a circular r-colouring f of G such that $f(v) \in L(v)$. We investigate, for a graph, the relations between its circular consecutive choosability and choosability. It is proved that for any positive integer k, if a graph G is k-choosable, then $ch_{cc}(G) \leqslant k + 1 - 1/k$; moreover, the bound is sharp for $k \geqslant 3$. For $k = 2$, it is proved that if G is 2-choosable then $ch_{cc}(G) \leqslant 2$, while the equality holds if and only if G contains a cycle. In addition, we prove that there exist circular consecutive 2-choosable graphs which are not 2-choosable. In particular, it is shown that $ch_{cc}(G) = 2$ holds for all cycles and for $K_{2,n}$ with $n \geqslant 2$. On the other hand, we prove that $ch_{cc}(G) > 2$ holds for many generalized theta graphs.

Keywords: choosability, circular consecutive choosability.

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1 Introduction

Choosability is a notion introduced independently by Vizing [9] in 1976 and Erdős, Rubin, and Taylor [2] in 1980, and has been widely studied ever since. Let G be a graph. A *list assignment* is a function L that assigns to each vertex with a set of permissible colours. We call G *list L-colourable* if there exists a proper colouring f such that $f(v) \in L(v)$ holds for every vertex v. A graph G is *k-choosable* if G is list L-colourable for every list L with $|L(v)| = k$ for all v. The *choice number* or *choosability* of G is defined as

$$ch(G) = \min\{k\colon G \text{ is } k\text{-choosable}\}.$$

Besides the choice number, several variations of choosability have also been studied in the literature. One of them is the consecutive choosability, introduced by Waters [10], in which the list assignment for each vertex is a set of consecutive integers. Another variation, called circular choosability, is motivated by the circular colouring of graphs.

For a positive real number r, let $S(r)$ denote the circle obtained from the interval $[0, r]$ by identifying 0 and r into a single point. For a real number t, denote by $[t]_r$ the remainder of t upon division by r. For $a, b \in S(r)$, the distance between a and b is $|a-b|_r = \min\{|a-b|, r-|a-b|\}$, and the intervals $[a, b]_r$ and $(a, b)_r$ are defined as $[a, b]_r = \{t \in S(r) : [t-a]_r \leqslant [b-a]_r\}$ and $(a, b)_r = \{t \in [0, r) : 0 < [t-a]_r < [b-a]_r\}$. Suppose $G = (V, E)$ is a graph. A *circular r-colouring* of G is a mapping $f : V(G) \to S(r)$ such that the inequality $|f(u) - f(v)|_r \geqslant 1$ holds for every edge uv of G. The *circular chromatic number* $\chi_c(G)$ of G is defined as the least r such that G admits a circular r-colouring. It is known that $\chi(G) = \lceil \chi_c(G) \rceil$ holds for every graph G. Thus, the circular chromatic number of G is a refinement of the chromatic number of G. Circular colouring has been studied extensively in the literature in the past two decades (see [12, 13] for surveys on this subject).

The concept of circular choosability of graphs was first studied in the article by Zhu [14]. Given a graph G and a positive real number r, an *r-circular colour-list assignment* for G is a function L that assigns to each vertex v a set $L(v)$ of disjoint union of closed intervals on $S(r)$. A *circular L-colouring* of G is a circular r-colouring f of G such that $f(v) \in L(v)$ for every vertex v. For a real $t \leqslant r$, if for each v, the sum of the lengths of the disjoint intervals in $L(v)$ is equal to t, then L is called a *(t, r)-circular colour-list* *assignment*. A graph G is *circular t-choosable* if G admits a circular L-colouring for any r and for any (t,r)-circular colour-list assignment L. The *circular choosability* $ch_c(G)$ of G (also known as the *circular choice number* or the *circular list chromatic number*) is defined as:

$$ch_c(G) = \inf\{t : \ G \text{ is circular } t\text{-choosable}\}.$$

Parallel to the investigation of the consecutive variation of choosability [10], it is natural to consider the consecutive variation of circular choosability in which the list of each vertex is a single closed interval on $S(r)$. This is a notion first introduced and studied by Lin et al. [5]. An *r-circular consecutive colour-list assignment* of G is a mapping L which assigns to each vertex v with a closed interval $L(v)$ on $S(r)$. If for every vertex v, the length of $L(v)$ is a fixed real t, then L is called a *(t,r)-circular consecutive colour-list assignment* of G. We call G *circular consecutive (t,r)-choosable* if G is circular L-colourable for every (t,r)-circular consecutive colour-list assignment L of G.

Notice that if $r < \chi_c(G)$, then G is not circular L-colourable for any (t,r)-circular colour-list assignment L. Therefore, we restrict our attention to only real numbers r with $r \geqslant \chi_c(G)$.

Definition 1. *Suppose $r \geqslant \chi_c(G)$. The circular consecutive choosability of G with respect to r is defined as*

$$ch^r_{cc}(G) = \inf\{t : G \text{ is circular consecutive } (t,r)\text{-choosable}\}.$$

The circular consecutive choosability of G is defined as

$$ch_{cc}(G) = \sup\{ch^r_{cc}(G) : r \geqslant \chi_c(G)\}.$$

Equivalently, $ch_{cc}(G)$ is the infimum t such that G is circular consecutive (t,r)-choosable for any $r \geqslant \chi_c(G)$.

Circular colouring of graphs can be used as a model for periodic scheduling problems. Let each vertex of a graph G represent a job. Each job needs to be carried out once in every period of length r, and it takes one unit of time to finish each job once. A pair of adjacent vertices in G represent two jobs that one of them needs to be finished before the other can start (that is, the unit times to complete the two jobs are disjoint). A circular r-colouring of the graph G is a scheduling satisfying all the requirements above. One may add another natural constraint to the scheduling problem, namely, the job represented by a vertex v can only be performed during a certain time interval $L(v)$ on a period of length r. Then such a scheduling corresponds to a consecutive circular list colouring of the graph.

An application of circular choosability of graphs is its use to the inductive proofs of circular colourability of graphs. To prove a graph G is circular r-colourable, one may find a circular r-colouring f of $G-H$ for some induced subgraph H of G (by inductive hypothesis), then extend f to a circular r-colouring for G. In the extension, the colours available to vertices of H are restricted by the already coloured vertices in $G-H$. Thus we are facing a circular list colouring problem. Such techniques have been used in the study of the circular chromatic number for planar graphs with large girth (cf. [1, 3, 4, 11]).

In the inductive proof described above, if each vertex x of H is adjacent to only one coloured vertex in $G-H$, then the set of available colours to x is a closed interval on $S(r)$. Therefore we are facing a circular consecutive list colouring problem for H.

A circular consecutive list colouring is a special case of a circular list colouring in which the list of permissible colours for each vertex is a single closed interval. We may also view a circular colouring as a special list circular colouring of G in which each vertex is given the whole circle $S(r)$ as the set of permissible colours. In this sense, circular consecutive list colouring of a graph lies in between circular colouring and list circular colouring. Hence, the parameter $ch_{cc}(G)$ is naturally closely related to $\chi_c(G)$, $ch_c(G)$, and $ch(G)$. In [5], it was shown that if G is a graph on n vertices, then

$$\chi(G)-1 \leqslant ch_{cc}(G) \leqslant 2\chi_c(G)(1-1/n)-1.$$

The values of $ch_{cc}(G)$ for complete graphs, trees, even cycles, and balanced complete bipartite graphs were determined; upper and lower bounds for $ch_{cc}(G)$ were given for some other graphs [5].

In this article, we investigate for a graph G, the relations between $ch_{cc}(G)$ and $ch(G)$. In Section 2, we prove that if G is k-choosable, then $ch_{cc}(G) \leqslant k+1-1/k$. We then show that the bound is tight for $k \geqslant 3$: For any $k \geqslant 3$ and for any $\epsilon > 0$, there is a k-choosable graph G with $ch_{cc}(G) > k+1-1/k-\epsilon$. For $k=2$, we improve this bound by showing, in Section 3, that every 2-choosable graph G has $ch_{cc}(G) \leqslant 2$. This bound is also tight as any 2-choosable graph G containing a cycle has $ch_{cc}(G)=2$. Although the result for $k = 2$ implies that every 2-choosable graph is circular consecutive 2-choosable, the converse of the statement is not true. In Section 4, we show that every odd cycle (which is not 2-choosable) is circular consecutive 2-choosable.

For positive integers a, b, and c, the *theta graph* $\theta_{a,b,c}$ is constructed by joining two vertices with three internally disjoint paths of lengths a, b, and c, respectively. The *heart* of a graph is obtained by sequentially deleting vertices of degree 1. A complete characterization of 2-choosable graphs is obtained in [2]:

Theorem 1. *[2] A connected graph G is 2-choosable if and only if the heart of G is K_1, an even cycle, or $\theta_{2,2,2n}$ for some $n \geqslant 1$.*

A characterization of circular consecutive 2-choosable graphs remains an open problem. Referring to Theorem 1, to further investigate this problem, it is natural to study the family of generalized theta graphs. A *generalized theta graph* $\theta_{k_1,k_2,\cdots,k_n}$ is obtained by joining two vertices by n internally disjoint paths of lengths $k_1, k_2, \cdots, k_n$. In Section 5, we prove that $ch_{cc}(\theta_{\underbrace{2,2,\cdots,2}_{n}}) = 2$ holds for any integer $n \geqslant 2$. On the other hand, we show that $ch_{cc}(\theta_{2,2,2,n}) > 2$ for $n \neq 2, 4, 6$.

2 k-choosable graphs

We establish an upper bound of $ch_{cc}(G)$ for a graph G, in terms of the choosability of G (Corollary 3). Then we prove that for every $k \geqslant 3$, there exist k-choosable graphs whose circular consecutive choosability is arbitrarily close to the upper bound (Theorem 6).

Lemma 2. *Let $k \geqslant 2$ be an integer and let G be a graph with $ch(G) = k$ then $ch^r_{cc}(G) \leqslant k + (k-1)(r - \lfloor r \rfloor)/\lfloor r \rfloor$ for every $r \geqslant \chi_c(G)$.*

Proof. Let $s = k+(k-1)(r-\lfloor r \rfloor)/\lfloor r \rfloor$, and let L be an s-circular consecutive list assignment of G with respect to r. For $l = 0, 1, \dots \lfloor r \rfloor - 1$ let $I_l = [lr/\lfloor r \rfloor, (l+1)r/\lfloor r \rfloor - 1]_r$ be an interval in $S(r)$. For every $v \in V(G)$ let $S(v) = \{j \mid I_j \cap L(v) \neq \emptyset\}$. Since $L(v)$ is an interval of length $s = k + (k-1)(r - \lfloor r \rfloor)/\lfloor r \rfloor$, it follows that $|S(v)| \geqslant k$. As $ch(G) = k$ it is possible to choose $k(v) \in S(v)$ for every $v \in V(G)$ so that $k(v) \neq k(w)$ for every $vw \in E(G)$. By the choice of $S(v)$ we can choose $f(v) \in I_{k(v)} \cap L(v)$ for every

$v \in V(G)$. It remains to note that for every $i, j \in \{0, 1, \ldots, \lfloor r \rfloor - 1\}$, $i \neq j$ and every $x \in I_i$ $y \in I_j$ we have $|x-y|_r \geqslant 1$ and therefore $|f(v)-f(w)|_r \geqslant 1$ for every $vw \in E(G)$. $\square$

Corollary 3. *Let $k \geqslant 2$ be an integer. If a graph G has list chromatic number k, then $ch_{cc}(G) \leqslant k + 1 - 1/k$.*

Proof. If $\chi_c(G) \leqslant r \leqslant k$ then $ch^r_{cc}(G) \leqslant r \leqslant k$. If $r \geqslant k$ then $ch^r_{cc}(G) \leqslant k + (k-1)(r - \lfloor r \rfloor)/\lfloor r \rfloor < k + (k-1)/k$ by Lemma 2. $\square$

We shall show that for $k \geqslant 3$, the upper bound given in Corollary 3 is tight. For this purpose, we need an alternate definition of $ch_{cc}(G)$ given in [5].

Given positive integers $p \geqslant 2q$, a *(p, q)-colouring* of a graph G is a mapping $f : V(G) \to \{0, 1, \cdots, p-1\}$ such that for any edge xy of G, $q \leqslant |f(x) - f(y)| \leqslant p - q$. For any integer a, $[a]_p$ denotes the remainder of a divided by p. For $a, b \in \{0, 1, \cdots, p-1\}$, the *circular integral interval* $[a, b]_p$ is defined as

$$[a, b]_p = \{a, a+1, a+2, \cdots, b\}.$$

Here the additions are modulo p. Suppose G is a graph and p, q are positive integers such that $p/q \geqslant \chi_c(G)$, and s is a positive integer. Let $l : V(G) \to \{0, 1, \cdots, p-1\}$ be a mapping. A (p, q)-colouring f of G is *compatible with* (l, s) if for any vertex x, $f(x) \in [l(x), l(x) + s - 1]_p$. We say a graph G is *circular consecutive (p, q)-s-choosable* if for any mapping $l : V(G) \to \{0, 1, \cdots, p-1\}$, G has a (p, q)-colouring f which is compatible with (l, s). We define the consecutive (p, q)-choosability of G as

$$\tau_{p,q}(G) = \min\{s : G \text{ is circular consecutive } (p, q)\text{-}s\text{-choosable}\}.$$

The following lemma is proved in [5].

Lemma 4. *For any graph G and for any $r = p/q \geqslant \chi_c(G)$,*

$$\tau_{p,q}(G) = \lfloor ch^r_{cc}(G) \cdot q \rfloor + 1.$$

Now we prove a technical lemma which is later used to lower bound the maximum circular consecutive choosability of graphs of fixed treewidth.

A graph G is called a *k-tree* if the vertices of G can be ordered as $v_1, v_2, \cdots, v_n$ in such a way that $\{v_1, v_2, \cdots, v_k\}$ induces a K_k, and for each $j \geqslant k+1$, the set $N^+(v_j) = \{v_i : i < j, v_i \sim v_j\}$ induces a K_k. The *treewidth* of a graph G is the minimum k such that G is a subgraph of a k-tree.

Lemma 5. *Let $k \geqslant 2, p$ and q be positive integers such that $p/q \geqslant k$, and let s be a positive integer. Suppose that every graph G with treewidth at most $k-1$ is circular consecutive (p,q)-s-choosable. Then there exists a non-empty family $\mathcal{S}$ of k-element subsets of $\{0,1,\ldots,p-1\}$ such that for every $S \in \mathcal{S}$ the following conditions hold:*

1. *for every distinct $x_1, x_2 \in S$ we have $q \leqslant |x_1 - x_2| \leqslant p-q$,*
2. *for every $X \subset S$ with $|X| = k-1$ and every $i \in \{0,1,\ldots,p-1\}$ there exists $S' \in \mathcal{S}$ such that $S' = X \cup \{x_0\}$ and $x_0 \in [i, i+s-1]_p$.*

Proof. For a graph H and a (p,q)-colouring f of H let $\mathcal{S}(H,f)$ denote the family of sets of colours of cliques of size k in H, and let $\xi(H,f) = |\mathcal{S}(H,f)|$. Suppose $l : V(H) \to \{0,1,\ldots,p-1\}$ and there is a (p,q)-colouring f of H compatible with (l,s). Then let

$$\xi(H,(l,s)) = \min\{\xi(H,f) : f \text{ is a } (p,q)\text{-colouring of } H \text{ compatible with } (l,s)\}.$$

Choose a graph G of treewidth at most $k-1$ and a map $l : V(G) \to \{0,1,\ldots,p-1\}$ so that $\xi(G,(l,s))$ is maximum over all graphs of treewidth at most $k-1$ and all maps l. Construct the graph G' and a map $l' : V(G') \to \{0,1,\ldots,p-1\}$ as follows: For every clique $W \subseteq V(G)$ with $|W| = k-1$ and every $i \in \{0,1,\ldots,p-1\}$ create a vertex v_W^i of degree $k-1$ of G' that is joined by edges to vertices of W and set $l'(v_W^i) = i$. Let l' be identical to l on $V(G)$. Then G' has treewidth at most $k-1$. By the choice of G there exists a (p,q)-colouring f' of G' compatible with (l',s) such that $\mathcal{S}(G',f') = \mathcal{S}(G,f')$.

We claim that $\mathcal{S} = \mathcal{S}(G,f')$ satisfies the requirements of the lemma. Clearly $\mathcal{S}$ is non-empty. For every $S \in \mathcal{S}$ there exists a clique $U \subseteq V(G)$ such that $S = f'(U)$. Therefore the first requirement is satisfied by the definition of a (p,q)-colouring. Similarly, for every $X \subset S$ with $|X| = k-1$ there exists a clique $W \subset U$ such that $|W| = k-1$ and $X = f'(W)$. Then $S' = f'(W \cup \{v_W^i\})$ satisfies the second requirement. $\square$

Theorem 6. *For every $k \geqslant 3$ and $\varepsilon > 0$ there exists a graph $G_{k,\varepsilon}$ such that $G_{k,\varepsilon}$ has treewidth at most $k-1$ and $ch_{cc}(G_{k,\varepsilon}) > k+1-1/k-\varepsilon$.*

Proof. We will show that for every positive integer n and every integer $k \geqslant 3$ there exists a graph $G_{k,n}$ of treewidth at most $k-1$ that is not circular consecutive (p,q)-s-choosable, where $p = nk(k+1)-2, q = nk$ and

$s = nk(k+1) - n - 2$. By Lemma 4, for $r = k + 1 - 2/nk$, $ch_{cc}(G_{k,n}) \geqslant ch^r_{cc}(G_{k,n}) > (p-1)/q = (nk(k+1) - n - 2 - 1)/nk = k + (k-1)/k - 3/nk$.

Suppose, on the contrary, that for some n and some $k \geqslant 3$ every graph of treewidth at most $k-1$ is circular consecutive (p,q)-s-choosable. By Lemma 5 then there exists a family $\mathcal{S}$ of k-element subsets of $\{0, 1, \ldots, p-1\}$ satisfying the requirements of that lemma.

Choose $S = \{a_1, \ldots, a_k\} \in \mathcal{S}$ so that $a_1, \ldots, a_k$ appear in $\{0, 1, \ldots, p-1\}$ in circular order and $([a_2 - a_1]_p, [a_3 - a_2]_p, \ldots, [a_k - a_{k-1}]_p)$ is lexicographically maximum. Let $a_{k+1} = a_1$, by convention.

Consider $X = S - \{a_2\}$ and $i = a_1 + \lceil ([a_3 - a_1]_p + n)/2 \rceil$. Then by condition 2 in Lemma 5 there exists $S' \in \mathcal{S}$ such that $S' = X \cup \{a'_2\}$ and $a'_2 \in [i, i + s - 1]$. Note that $a'_2 \in [a_1, a_3]_p$. Otherwise $a'_2 \in [a_l, a_{l+1}]_p$ for some $l \geqslant 3$, and we obtain contradiction as follows,

$$\begin{aligned} p &= \sum_{j=1}^{l-1} [a_{j+1} - a_j]_p + [a'_2 - a_l]_p + [a_{l+1} - a'_2]_p + \sum_{j=l+1}^{k} [a_{j+1} - a_j]_p \\ &\geqslant q(l - 1 + 2 + k - l) = q(k+1) = p + 2. \end{aligned}$$

Since $a'_2 \notin [i - n, i - 1]$, it follows that

$$|[a_3 - a'_2]_p - [a'_2 - a_1]_p| \geqslant n - 1.$$

Hence

$$\max\{[a_3 - a'_2]_p, [a'_2 - a_1]_p\} \geqslant ([a_3 - a'_2]_p + [a'_2 - a_1]_p + n - 1)/2 = ([a_3 - a_1]_p + n - 1)/2.$$

By the choice of S, we have

$$\begin{aligned} [a_2 - a_1]_p &\geqslant \max\{[a'_2 - a_1]_p, [a_3 - a'_2]_p\} \\ &\geqslant ([a_3 - a_1]_p + n - 1)/2 = ([a_3 - a_2]_p + [a_2 - a_1]_p + n - 1)/2. \end{aligned}$$

Consequently,

$$[a_2 - a_1]_p \geqslant [a_3 - a_2]_p + n - 1.$$

By considering $X = S - \{a_l\}$ for $l \in \{3, \ldots, k\}$ and $i = a_{l-1} + q + n$, and using an argument similar to the above, we deduce $[a_l - a_{l-1}]_p \geqslant q + n$. A contradiction follows:

$$\begin{aligned} p &= \sum_{j=1}^{k} [a_{j+1} - a_j]_p \geqslant [a_2 - a_1]_p + (q+n)(k-2) + [a_1 - a_k]_p \\ &\geqslant q + n + n - 1 + (q+n)(k-2) + q = (q+n)k - 1 = p + 1. \end{aligned}$$

$\square$

Since graphs of treewidth at most $(k-1)$ are k-choosable, Theorem 6 shows that the bound of Corollary 3 is tight.

Corollary 7. *If G is a series-parallel graph, then $ch_{cc}(G) \leqslant 11/3$. For any $\epsilon > 0$, there is a series-parallel graph G with $ch_{cc}(G) > 11/3 - \epsilon$.*

3 2-choosable graphs

We improve the bound in Corollary 3 for $k = 2$. Precisely, we prove that if G is 2-choosable, then $ch_{cc}(G) \leqslant 2$. Combining this with a result in [5], the value of $ch_{cc}(G)$ can be determined in linear time for any 2-choosable graph G.

It is easy to see that G is circular consecutive 2-choosable if and only if the heart of G is circular consecutive 2-choosable. The graphs K_1 and even cycles are known [5] to be circular consecutive 2-choosable. To prove that every 2-choosable graph is circular consecutive 2-choosable, by Theorem 1 it remains to show that for any positive integer n, $ch_{cc}(\theta_{2,2,2n}) = 2$, which we prove in the following result.

Theorem 8. *Let $G = \theta_{2,2,2n}$ with $V(G) = \{u, v, x_1, x_2, \cdots, x_{2n+1}\}$ and $E(G) = \{x_1u, x_1v, x_{2n+1}u, x_{2n+1}v\} \cup \{x_jx_{j+1} : j = 1, 2, \cdots, 2n\}$. Let $r \geqslant 2$ and $l : V(G) \to S(r)$ be an arbitrary mapping and let $L(x) = [l(x), l(x)+2]_r$. Then G is circular L-colourable.*

To prove Theorem 8, we first establish a lemma concerning circular list colouring of paths. Given a path $P = (p_0, p_1, \ldots, p_k)$ and a list-assignment L that assigns to each vertex of P an interval of $S(r)$, we want to find possible colours that can be assigned to p_0 and p_k in a circular L-colouring of P.

Theorem 9. *Suppose $2 < r < 4$ and $k \geqslant \lceil 2/(r-2) \rceil$ and $P = (p_0, p_1, \cdots, p_k)$ is a path of length k. Let $l : P \to S(r)$ be any mapping such that $|l(p_i) - l(p_{i+1})|_r \geqslant 1$ for $0 \leqslant i \leqslant k-1$. Let $L(p_i) = [l(p_i), l(p_i) + 2]_r$. Then the following hold:*

(1) There exists a point $t \in L(p_0)$ such that for any $t' \in L(p_k)$ there is a circular L-colouring f of P with $f(p_0) = t$ and $f(p_k) = t'$.

(2) For any $0 < \ell < 2$, there exist an interval $X \subseteq L(p_0)$ of length ℓ and an interval $Y \subseteq L(p_k)$ of length $2 - \ell$, such that for any $t \in X$ and for

any $t' \in Y$ there is a circular L-colouring f of P with $f(p_0) = t$ and $f(p_k) = t'$.

By taking ℓ to be real number approaching 0, we can view statement (1) as a limit case of statement (2), where a single colour is viewed as an colour interval of length 0. Nevertheless, we shall prove the two statements separately.

To prove Theorem 9, we first define some notation and present two lemmas. We say two colours $t, t' \in S(r)$ are *adjacent* if $|t-t'|_r \geqslant 1$. For $t \in S(r)$, denote by $N(t)$ the set of colours adjacent to t, namely $N(t) = [t+1, t-1]_r$. For a subset A of $S(r)$, let $N(A) = \cup_{t \in A} N(t)$.

Lemma 10. *Suppose $I = [a, b]_r$ is an interval of $S(r)$ of length $\ell = [b-a]_r$. If $\ell \geqslant 2$, then $N(I) = S(r)$. Otherwise $N(I) = [a+1, b-1]_r$.*

The proof of Lemma 10 is trivial and omitted.

Lemma 11. *Suppose $2 < r < 4$, $a, b \in S(r)$ and $|a-b|_r \geqslant 1$. If $I = [s, s+\ell]_r \subseteq [b, b+2]_r$, then the following hold.*

(1) If $\ell \geqslant r-2$, then there is an interval I' of length $\ell - (r-2)$ such that $I' \subseteq [a, a+2]_r$ and $I = N(I')$.

(2) If $\ell \leqslant r-2$, then there is a colour $t' \in [a, a+2]_r$ such that $I \subseteq N(t')$.

Proof. First we observe that if $\ell = r-2$, then by (1), there is an interval I' of $[a, a+2]_r$ of length 0 such that $I = N(I')$. Here by an interval of length 0 we mean a single point. So in this case, the conclusions in (1) and (2) coincide.

(1): Assume $\ell \geqslant r-2$. Let $I' = [s-1, s-1+\ell-(r-2)]_r$. By Lemma 10, $N(I') = I = [s, s+\ell]_r = I$. Now we show that $I' \subseteq [a, a+2]_r$.

First we show that

$$I' = [s-1, s-1+\ell-(r-2)]_r \subseteq [b-1, b+3]_r.$$

Assume $t \in [s-1, s-1+\ell-(r-2)]_r$. Then $[t-(s-1)]_r \leqslant \ell-(r-2)$. We need to show that $[t-(b-1)]_r \leqslant [b+3-(b-1)]_r = 4-r$. Observe that

$$[t-(b-1)]_r = [t-(s-1)+s-b]_r = [[t-s+1]_r + [s-b]_r]_r.$$

Because $[s, s+\ell]_r \subseteq [b, b+2]_r$, we conclude that $[s-b]_r \leqslant 2-\ell$. Hence

$$[[t-s+1]_r + [s-b]_r]_r = [t-s+1]_r + [s-b]_r \leqslant \ell-(r-2)+2-\ell = 4-r.$$

It remains to show that $[b-1,b+3]_r \subseteq [a,a+2]_r$. If $t \in [b-1,b+3]_r$, then $[t-b+1]_r \leqslant 4-r$. Because $1 \leqslant [b-a]_r \leqslant r-1$, we have $[b-1-a]_r \leqslant r-2$. It follows that

$$\begin{aligned}[t-a]_r &= [t-b+1+b-1-a]_r \\ &= [[t-b+1]_r + [b-1-a]_r]_r \leqslant 4-r+r-2 = 2.\end{aligned}$$

Therefore $t \in [a, a+2]_r$.

(2): Assume $\ell \leqslant r-2$. Let I'' be an interval contained in $[b, b+2]_r$ of length $r-2$ such that $I \subseteq I''$. Apply (1) to I'', we conclude that there exists $t \in [a, a+2]_r$ such that $I'' = N(t)$. Hence $I \subseteq N(t)$. □

Proof of Theorem 9 We first consider the case that $k = \lceil 2/(r-2) \rceil$.

(1) Let $I_k = L(p_k)$. By repeatedly applying Lemma 11, we conclude that there are intervals $I_{k-1}, I_{k-2}, \cdots, I_1$ such that

- I_j is contained in $L(p_j)$.
- I_j has length $2-(k-j)(r-2)$.
- $I_{j+1} = N(I_j)$.

Since I_1 has length $2-(k-1)(r-2) \leqslant r-2$, apply Lemma 11 again, there is a colour $t \in L(p_0)$ such that $I_1 \subseteq N(t)$.

For any $t' \in L(p_k) = I_k$, there are colours $c_j \in I_j$ for $j = k-1, k-2, \cdots, 1$ such that $t' \in N(c_{k-1})$ and $c_{j+1} \in N(c_j)$ for $j = k-2, k-3, \cdots, 1$ and $c_1 \in N(t)$. Let $f(p_0) = t, f(p_k) = t'$ and $f(p_j) = c_j$ for $j = 1, 2, \cdots, k-1$. Then f is a circular L-colouring of P_k satisfying the requirements of the theorem. This completes the proof of (1).

(2) Let $q = \lceil \ell/(r-2) \rceil$. Similarly as in the proof of (1), by repeatedly applying Lemma 11, we have the following:

- For $j = k, k-1, k-2, \cdots, q$, there are intervals $I_j \subseteq L(p_j)$ of length $2-(k-j)(r-2)$ and $I_{j+1} = N(I_j)$ for $j = k-1, k-2, \cdots, q$.
- For $j = 0, 1, 2, \cdots, q$, there are intervals $J_j \subseteq L(p_j)$ of length $2-j(r-2)$ with $N(J_j) = J_{j-1}$ for $j = 1, 2, \cdots, q$.

Let

$$\delta = q(r-2) - \ell \text{ and } \epsilon = (k-q)(r-2) + \ell - 2.$$

Let J'_q be a closed interval contained in $L(p_q)$ of length $2 - q(r-2) + \delta$ containing J_q, and let I'_q be a closed interval contained in $L(p_q)$ of length $2-(k-q)(r-2)+\epsilon$ containing I_q. As the sum of the lengths of J'_q and I'_q is equal to 2 and both are contained in $L(p_q)$ which is an interval of length 2, $I'_q \cap J'_q \neq \emptyset$.

Let $s \in I'_q \cap J'_q$. Since $I_q \subseteq I'_q$ and I_q has length $2-(k-q)(r-2)$, there is a colour $s' \in I_q$ such that $|s-s'|_r \leqslant \epsilon$. Thus $N(s)$ is an interval which is a shift of the interval $N(s')$ by a distance $|s-s'|_r \leqslant \epsilon$. Since $N(s') \cap I_{q+1} = N(s')$, which is an interval of length $r-2$, it follows that $I'_{q+1} = N(s) \cap I_{q+1}$ is an interval of length at least $r-2-\epsilon$. For $j = q+2, q+3, \cdots, k$, let $I'_j = N(I'_{j-1})$, then $I'_j \subseteq I_j \subseteq L(p_j)$ and has length at least $(j-q)(r-2)-\epsilon$. In particular, $I'_k \subseteq L(p_k)$ has length at least $(k-q)(r-2)-\epsilon = 2-\ell$. Similarly, let $J'_{q-1} = N(s) \cap J_{q-1}$ for $j = q-2, q-3, \cdots, 1$, let $J'_j = N(J'_{j+1})$. We have $J'_j \subseteq L(p_j)$ and J'_0 has length $q(r-2)-\delta = \ell$.

Let $X = J'_0$ and $Y = I'_k$. For $t \in X$ and $t' \in Y$, there are colours $c_j \in I'_j$ for $j = k-1, k-2, \cdots, q+1$ such that $t' \in N(c_{k-1})$ and $c_j \in N(c_{j-1})$ for $j = k-1, k-2, \cdots, q+1$. Similarly, there are colours $c_j \in J'_j$ for $j = 1, 2, \cdots, q-1$ such that $t \in N(c_1)$ and $c_{j+1} \in N(c_j)$ for $j = 1, 2, \cdots, q-2$. Then $f(p_k) = t'$, $f(p_0) = t$, $f(p_q) = s$ and $f(p_j) = c_j$ for $j = 1, 2, \cdots, q-1, q+1, \cdots, k-1$ is a circular L-colouring f with $f(p_0) = t$ and $f(p_k) = t'$.

Assume Theorem 9 holds for k. To prove that it also holds for $P_{k+1} = (p_0, p_1, \cdots, p_k, p_{k+1})$, we apply the theorem to the path $(p_0, p_1, \cdots, p_k)$ to obtain the required sets X and Y, and then let $Y' = Y + l(p_{k+1}) - l(p_k) = \{t + l(p_{k+1}) - l(p_k) : t \in Y\}$. Then X, Y' are the required sets for statement (2). Statement (1) is proved in the same way. □

Now we are ready to prove Theorem 8. Assume Theorem 8 is not true. Let n be the smallest integer for which there is a real number $r \geqslant 2$, a $(2,r)$-circular consecutive colour-list assignment L of G, such that G is not circular L-colourable. We shall derive some properties of the list assignment L that eventually lead to a contradiction.

It is known [5] that we only need to consider those r with $2 \leqslant r < 4$. As $\theta_{2,2,2} = K_{2,3}$, we know that $\theta_{2,2,2}$ is circular consecutive 2-choosable (see next section for a proof showing that $K_{2,n}$ are circular consecutive 2-choosable). In the following, we assume that $2 \leqslant r < 4$ and $n \geqslant 2$.

If $r \leqslant 2 + 2/n$, then $L(x_1) \cap L(x_3) \cap \cdots \cap L(x_{2n+1}) \neq \emptyset$. Let $t \in L(x_1) \cap L(x_3) \cap \cdots L(x_{2n+1})$. Let $f(x_{2j+1}) = t$ for $j = 0, 1, \cdots, n$. For

$w \in \{u, v, x_2, x_4, \cdots, x_{2n}\}$, let $f(w)$ be any colour from the nonempty set $L(w) - (t-1, t+1)_r$. Then f is an L-colouring of G. In the following, we assume that $r > 2 + 2/n$.

Lemma 12. *For any $j \in \{2, 3, \cdots, 2n-1\}$, $l(x_j)$ and $l(x_{j+1})$ are adjacent, i.e., $|l(x_j) - l(x_{j+1})|_r \geqslant 1$.*

Proof. Assume to the contrary that there exists an index $j \in \{2, 3, \cdots, 2n-1\}$ such that $|l(x_j) - l(x_{j+1})|_r < 1$. Delete two vertices x_j, x_{j+1} and add an edge $x_{j-1}x_{j+2}$. The resulting graph G' is $\theta_{2,2,2(n-1)}$. By the minimality of G, there exists a circular L-colouring f for G'. We shall extend f to a circular L-colouring of G, by finding appropriate colours for x_j and x_{j+1}.

Let $a = f(x_{j-1})$ and $b = f(x_{j+2})$. If $b \in L(x_j)$, then let $f(x_j) = b$ and let $f(x_{j+1})$ be any colour from the non-empty set $L(x_{j+1}) - (b-1, b+1)_r$. Then f is a circular L-colouring of G. Thus we may assume that $b \notin L(x_j)$. Similarly, we may assume that $a \notin L(x_{j+1})$.

Since $r < 4$, either $a + 1 \in L(x_j)$ or $a - 1 \in L(x_j)$. By symmetry, we may assume that $a + 1 \in L(x_j)$. Similarly, either $b + 1 \in L(x_{j+1})$ or $b - 1 \in L(x_{j+1})$. If $b+1 \in L(x_{j+1})$, then let $f(x_j) = a+1$ and $f(x_{j+1}) = b+1$. Then f is a circular L-colouring of G. Thus we may assume that $b + 1 \notin L(x_{j+1})$ and hence $b-1 \in L(x_{j+1})$. Moreover, we may also assume that $a-1 \notin L(x_j)$, for otherwise, by letting $f(x_j) = a - 1$ and $f(x_{j+1}) = b - 1$ we obtain a circular L-colouring of G. Let $f(x_j) = l(x_j) + 2$ and $f(x_{j+1}) = l(x_j) + 1$. We shall show that f is a circular L-colouring of G.

Since $a + 1 \in L(x_j)$ and $a - 1 \notin L(x_j)$, it follows that $[a - 1, a]_r \subseteq [l(x_j)+2, a]_r$ and $[a, a+1]_r \subseteq [a, l(x_j)+2]_r$. Hence $[a-(l(x_j)+2)]_r \geqslant 1$ and $[l(x_j)+2-a]_r \geqslant 1$. I.e., $|f(x_j) - f(x_{j-1})|_r \geqslant 1$. Since $|l(x_j) - l(x_{j+1})|_r < 1$, it follows that $l(x_j) + 1 \in L(x_{j+1})$. I.e., $f(x_{j+1}) \in L(x_{j+1})$. By definition, $|f(x_j) - f(x_{j+1})|_r = 1$. Since $b \notin L(x_j)$, we have $|b - (l(x_j) + 1)|_r \geqslant 1$. I.e., $|f(x_{j+1}) - f(x_{j+2})|_r \geqslant 1$. This proves that f is indeed a circular L-colouring of G. $\square$

Let $l(x_1) = a$, $l(x_{2n+1}) = b$, $l(u) = c$ and $l(v) = d$. Without loss of generality, we may assume that

$$c \in L(v) = [d, d+2]_r.$$

Lemma 13. *Under the above assumption, we have $d \notin [c, c+2]_r$.*

Proof. Assume to the contrary that $c \in [d, d+2]_r$ and $d \in [c, c+2]_r$. By our assumption, $r \geqslant 2 + 2/n$. By Theorem 9, there is a colour $t \in L(x_2)$ such that for any $t' \in L(x_{2n})$, there is a circular L-colouring f of the path $(x_2, x_3, \cdots, x_{2n})$ with $f(x_2) = t$ and $f(x_{2n}) = t'$.

We construct a circular L-colouring c of G as follows: Let $c(x_2) = t$, and let $c(x_1) \in L(x_1)$ be any colour adjacent to t. Since $c \in [d, d+2]_r$ and $d \in [c, c+2]_r$, we have

$$[c, c+2]_r \cap [d, d+2]_r = [c, d+2]_r \cup [d, c+2]_r.$$

As $N([c, d+2]_r) = [c+1, d+1]_r$ and $N([d, c+2]_r) = [d+1, c+1]_r$, it implies that

$$N([c, d+2]_r \cup [d, c+2]_r) = S(r).$$

In particular, $c(x_1) \in N([c, c+2]_r \cap [d, d+2]_r)$. Let $s \in [c, c+2]_r \cap [d, d+2]_r$ be a colour adjacent to t and let $t^* \in L(x_{2n+1})$ be any colour adjacent to s. Let $c(u) = c(v) = s$ and let $c(x_{2n+1}) = t^*$. Let $t' \in L(x_{2n})$ be any colour adjacent to t^*. By the previous paragraph, c can be extended to a circular L-colouring of the path $(x_2, x_3, \cdots, x_{2n})$. $\square$

Lemma 14. $N([c, d+2]_r) \cup (N(c+2) \cap N(d)) = S(r)$.

Proof. By definition, $N([c, d+2]_r) = [c+1, d+1]_r$. Since $d \notin [c, c+2]_r$, we have $N(c+2) \cap N(d) = [c+3, c+1]_r \cap [d+1, d-1]_r = [d+1, c+1]_r$. $\square$

Proof of Theorem 8 Assume first that $b \notin (c+1, d-1)_r$. By Theorem 9, there is a colour $t \in L(x_1)$ such that for any $t' \in L(x_{2n+1})$, there is a circular L-colouring f of the path $(x_1, x_2, x_3, \cdots, x_{2n+1})$ with $f(x_1) = t$ and $f(x_{2n+1}) = t'$. We construct a circular L-colouring c of G as follows: Let $c(x_1) = t$. If $[c, d+2]_r \cap N(t) \neq \emptyset$ then let $c(u) = c(v) = s$ for some $s \in [c, d+2]_r \cap N(t)$, let $c(x_{2n+1}) = t'$, where $t' \in L(x_{2n+1})$ is any colour adjacent to s. By the choice of t, c can be extended to a circular L-colouring of the path $(x_1, x_2, x_3, \cdots, x_{2n+1})$. If $[c, d+2]_r \cap N(t) = \emptyset$, then $t \notin N([c, d+2]_r)$. By Lemma 14, t is adjacent to both $c+2$ and d. In this case, let $c(u) = c+2, c(v) = d$. Since $b \notin (c+1, d-1)_r$, it follows that $[b, b+2]_r \cap [d+1, c+1]_r \neq \emptyset$. Let $t' \in [b, b+2]_r \cap [d+1, c+1]_r$. Then t' is adjacent to both $c+2$ and d. Let $c(x_{2n+1}) = t'$. By the choice of t, c can be extended to a circular L-colouring of the path $(x_1, x_2, x_3, \cdots, x_{2n+1})$.

Assume next that $b \in (c+1, d-1)_r$. Then $[b, b+2]_r \cap (d+1, c+1)_r = \emptyset$. This implies that for any $t \in [b, b+2]_r$, $N(t) \cap [c, d+2]_r \neq \emptyset$. By Theorem

9, there is a colour $t \in L(x_{2n+1})$ such that for any $t' \in L(x_1)$, there is a circular L-colouring f of the path $(x_1, x_2, x_3, \cdots, x_{2n})$ with $f(x_1) = t'$ and $f(x_{2n+1}) = t$.

Let $s \in [c, d+2]_r \cap N(t)$ be a colour adjacent to t and let $t' \in L(x_1)$ be any colour adjacent to s. Let $c(u) = c(v) = s$ and let $c(x_1) = t'$ and $c(x_{2n+1}) = t$. Then c can be extended to a circular L-colouring of the path $(x_1, x_2, x_3, \cdots, x_{2n+1})$. This completes the proof of Theorem 8. $\square$

It is known [5] that if G contains a cycle, then $ch_{cc}(G) \geqslant 2$. If G is an n-vertex tree, then $ch_{cc}(G) = 2(1 - \frac{1}{n})$. Hence, for a connected 2-choosable graph G on n vertices, to determine the exact value of $ch_{cc}(G)$ it suffices to determine whether G contains a cycle or not. That is, if G contains a cycle then $ch_{cc}(G) = 2$; otherwise, $ch_{cc}(G) = 2 - \frac{2}{n}$. In conclusion, for 2-choosable graphs G, $ch_{cc}(G)$ can be determined in linear time.

4 Cycles

Although every 2-choosable graph is circular consecutive 2-choosable, the converse is not true. For example, by Theorem 1, $K_{2,n}$ is not 2-choosable for $n \geqslant 4$. However, it is easy to show that $K_{2,n}$ is circular consecutive 2-choosable: We only need to consider r with $2 \leqslant r < 4$. Denote $V(K_{2,n}) = \{u, v\} \cup \{x_1, x_2, \ldots, x_n\}$. Let L be a $(2, r)$-circular consecutive colour-list assignment for $K_{2,n}$. Then $L(u) \cap L(v) \neq \emptyset$. Let $f(u) = f(v) = t \in L(u) \cap L(v)$, and let $f(x_i) \in L(x_i) \setminus (t-1, t+1)_r$. Then f is a circular L-colouring of $K_{2,n}$.

The following is an open problem:

Question 1. *Which are the graphs G with $ch_{cc}(G) \leqslant 2$?*

As discussed in the previous paragraph, to investigate Question 1 it suffices to consider graphs without vertices of degree 1. So far, there are only two families of graphs that are known to have a positive answer to Question 1. Besides $K_{2,n}$ discussed in the previous paragraph, cycles is the other known family of such graphs, which we prove in the next result.

Theorem 15. *For any integer $n \geqslant 3$, $ch_{cc}(C_n) = 2$.*

The rest of this section is devoted to the proof of Theorem 15. It is proved in [5] that for any $n \geqslant 3$, $ch_{cc}(C_n) \geqslant 2$, and if n is even or $n = 3$ then the equality holds. To prove Theorem 15, it suffices to show that for any $k \geqslant 2$, for any $r \geqslant 2 + 1/k$, $ch^r_{cc}(C_{2k+1}) \leqslant 2$. To this end, the following lemma is established. Let L be a 2-circular-consecutive-list assignment for C_n with respect to r, where $V(C_n) = \{v_0, v_2, \cdots, v_{n-1}\}$, $v_i \sim v_{i+1}$. We need to find a circular L-colouring for C_n. Let $L(v_i) = [a_i, a_i + 2]_r$.

Lemma 16. *If $r \geqslant \chi_c(C_{n-2})$ and $ch^r_{cc}(C_{n-2}) \leqslant 2$, then $ch^r_{cc}(C_n) \leqslant 2$.*

Proof. Assume $ch^r_{cc}(C_{n-2}) \leqslant 2$ for some $r \geqslant \chi_c(C_{n-2})$. If $|a_i - a_{i+1}|_r \geqslant 1$ for all i, then $f(v_i) = a_i$, $0 \leqslant i \leqslant n-1$, is a circular L-colouring for C_n. Hence, assume without loss of generality, $a_{n-1} = 0$ and $a_0 = \varepsilon$ for some $0 \leqslant \varepsilon < 1$. Let G' be an $(n-2)$-cycle with the vertex set $\{v_1, v_2, \cdots, v_{n-2}\}$ where $v_1 v_{n-2}, v_i v_{i+1} \in E(G')$ for $i = 1, 2, \cdots, n-3$. Restricting L to $V(G')$ is indeed a 2-circular-consecutive-list assignment for C_{n-2} with respect to r. Since $ch^r_{cc}(C_{n-2}) \leqslant 2$, there exists a circular L-colouring f for G'. It suffices to extend f to C_n by finding $f(v_0) \in L(v_0)$ and $f(v_{n-1}) \in L(v_{n-1})$ such that $|f(v_0) - f(v_{n-1})|_r \geqslant 1$, $|f(v_0) - f(v_1)|_r \geqslant 1$ and $|f(v_{n-2}) - f(v_{n-1})|_r \geqslant 1$.

Suppose $f(v_{n-2}) \in L(v_0)$. Let $f(v_0) = f(v_{n-2})$. Since $L(v_{n-1})$ has length 2, there exists $j \in L(v_{n-1})$ with $|j - f(v_{n-2})|_r \geqslant 1$. Let $f(v_{n-1}) = j$. Then f is a circular L-colouring for C_n. Similarly, the result follows if $f(v_1) \in L(v_{n-1})$.

It remains to consider that $f(v_{n-2}) \notin L(v_0)$ and $f(v_1) \notin L(v_{n-1})$. Denote $x_1 = f(v_1) + 1$, $x_2 = f(v_1) - 1$, $y_1 = f(v_{n-2}) + 1$ and $y_2 = f(v_{n-2}) - 1$. Then $|x_i - y_i|_r \geqslant 1$ for $i = 1, 2$. If $x_1, x_2 \notin L(v_0)$, the result follows by letting $f(v_0) = \varepsilon$ and $f(v_{n-1}) = 1 + \varepsilon$. Similarly, it holds if $y_1, y_2 \notin L(v_{n-1})$. If $x_1 \in L(v_0)$ and $y_1 \in L(v_{n-1})$, we let $f(v_0) = x_1$ and $f(v_{n-1}) = y_1$. By symmetry, it remains to consider that $x_1 \in L(v_0)$, $y_1 \notin L(v_{n-1})$, $x_2 \notin L(v_0)$ and $y_2 \in L(v_{n-1})$. For this case, let $f(v_0) = 2$ and $f(v_{n-1}) = 1$. It is straightforward to check that each f defined above is a circular L-colouring for C_n. We leave the details to the readers. This completes the proof of Lemma 16. □

It is known [5] that for $r \geqslant 3$, $ch^r_{cc}(C_3) = 2$. To complete the proof of Theorem 15, by Lemma 16, it remains to show that if $n = 2k + 1 \geqslant 5$ and $2 + \frac{1}{k} \leqslant r < 2 + \frac{1}{k-1}$, then $ch^r_{cc}(C_n) \leqslant 2$. We prove this result in Lemma 18, below. To provide the reader with a better intuition on the seemingly complicated proof of Lemma 18, we shall prove in Lemma 17 a special case of this result. Indeed, the main idea of the proof of Lemma 18 stems from the proof of Lemma 17.

Lemma 17. *If $r = \chi_c(C_n)$, then $ch^r_{cc}(C_n) \leqslant 2$.*

Proof. Assume $n = 2k+1$ and $r = 2 + \frac{1}{k}$. Let $z = \frac{r}{n} = \frac{1}{k}$. Without loss of generality, assume $L(v_0) = [0,2]_r$. For $v \in C_n$, let $\overline{L}(v) = \{j : jz \in L(v)\}$. Then $\overline{L}(v_0) = \{0, 1, \ldots, n-1\}$. For each other vertex v, $\overline{L}(v)$ is a set of circular consecutive integers, modulo n, from $\{0, 1, 2, \ldots, n-1\}$. Furthermore, there is at most one i, $0 \leqslant i \leqslant n-1$, such that $i \notin \overline{L}(v)$. For $i = 0, 1, \ldots, n-1$, let

$$\phi_i(v_j) = (i + jk) \bmod n.$$

Then for each i, ϕ_i is a $(2k+1, k)$-colouring of C_n. For each vertex v, there is at most one i such that $\phi_i(v) \notin \overline{L}(v)$, while for v_0, $\phi_i(v_0) \in \overline{L}(v_0)$ for all i. Thus there is an index i^* such that $\phi_{i^*}(v) \in \overline{L}(v)$ for all vertices v of C_n. Letting $f(v) = \phi_{i^*}(v)z$, we obtain a circular L-colouring for C_n. $\square$

Lemma 18. *If $2 + \frac{1}{k} \leqslant r < 2 + \frac{1}{k-1}$, then $ch^r_{cc}(C_n) \leqslant 2$.*

Proof. Let $z = \frac{r}{n}$. In the case $r = \frac{2k+1}{k}$, the colours used to colour the vertices of C_n are restricted to the set $\{iz : i = 0, 1, \ldots, n-1\}$. For $2 + \frac{1}{k} < r < 2 + \frac{1}{k-1}$, instead of restricting to this colour set, we restrict to colours in $\cup_{i=0}^{n-1} X_i$, where

$$X_i = [iz - x, iz]_r,$$

are intervals of length x ending at iz. Throughout the proof, when we encounter a negative real number, say w, on $S(r)$, we regard it as $r + w$ on $S(r)$. For instance, $X_0 = [r-x, 0]_r$. We choose the length x to be $(n-1)z-2$, which is the smallest real number such that every interval of $S(r)$ of length 2 intersects at least $n-1$ of the intervals X_i $(i = 0, 1, \ldots, n-1)$. Figure 1 illustrates the intervals X_i on $S(r)$. Note that when $r = 2 + \frac{1}{k}$, then $x = 0$ and each X_i is a single point of $S(r)$, $X_i = \{i/k\}$.

For each vertex v, denote $\overline{L}(v) = \{j : X_j \cap L(v) \neq \emptyset\}$. By our assumption, $L(v_0) = [0,2]_r$. Thus $\overline{L}(v_0) = \{0, 1, 2, \cdots, n-1\}$. For each v, $\overline{L}(v)$ contains at least $n-1$ of the integers $0, 1, \ldots, n-1$. Denote the missing number of $\overline{L}(v)$, if exists, by $m(v)$; otherwise $m(v) = \infty$. Observe,

$$m(v) = \begin{cases} i, & \text{if } \overline{L}(v) = \{i+1, i+2, \cdots, i-1\} \pmod{n}; \\ \\ \infty, & \text{if } L(v) = [jz, jz+2]_r \text{ for some } j \in \{0, 1, \cdots, n-1\}. \end{cases}$$

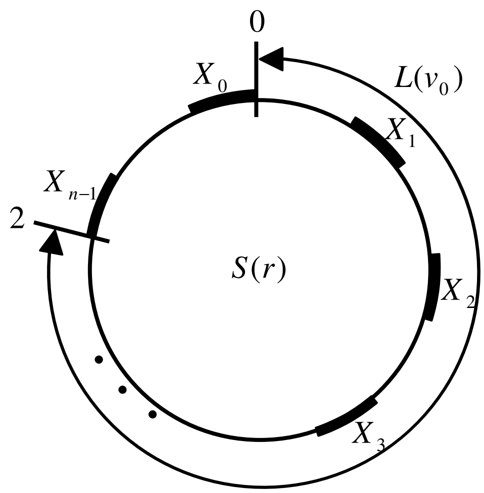

Figure 1: Locations of the intervals X_i's on $S(r)$, in the proof of Lemma 18

Let ϕ_i be defined as in the proof of Lemma 17. By the proof of Lemma 17, we can find an index $i \in \{0, 1, \ldots, n-1\}$ such that $\phi_i(v) \in \overline{L}(v)$ for all vertices v of C_n. To obtain a circular L-colouring f of C_n, we may let $f(v)$ be a colour from the set $X_{\phi_i(v)} \cap L(v)$. However, for some choices of the colours for $f(v)$ from $X_{\phi_i(v)} \cap L(v)$, the resulting mapping f may not be a circular r-colouring of C_n. For example, if $\phi_i(v_j) = a, \phi_i(v_{j+1}) = a + k$, $X_a \cap L(v_j) \subseteq (az - x/2, az]_r$ and $X_{a+k} \cap L(v_{j+1}) \subseteq [(a+k)z - x, (a+k)z - x/2)_r$, then straightforward calculation shows that for any $s \in X_a \cap L(v_j)$ and $s' \in X_{a+k} \cap L(v_{j+1})$, we have $|s - s'|_r < 1$. On the other hand, by a straightforward calculation, it can be verified that if s is the middle point of X_a, i.e., $s = az - \frac{1}{2}x$, then for any $s' \in X_{a+k}$, one has $|s - s'|_r \geqslant 1$.

Therefore, we need to take special care of the case when $L(v)$ intersects $X_{\phi_i(v)}$ but does not contain the middle point of $X_{\phi_i(v)}$. If $m(v) = i$ for some $i = 0, 1, 2, \cdots, n-1$, then $L(v)$ partly intersects X_{i+1} and X_{i-1}. Let $\alpha(v) = |L(v) \cap X_{i+1}|_r$ and $\beta(v) = |L(v) \cap X_{i-1}|_r$. If $m(v) = \infty$, let $\alpha(v) = \beta(v) = 0$. Referring to Figure 1, it is obvious that for any vertex v, $\alpha(v) + \beta(v) \geqslant x$ and hence $\alpha(v)$ and $\beta(v)$ cannot be both less than $x/2$, except when $m(v) = \infty$ in which $\alpha(v) = \beta(v) = 0$.

For $i = 0, 1, 2, \cdots, n-1$, set

$$\begin{aligned} \phi_i(v_j) &= (i + jk) \bmod n; \\ \psi_i(v_j) &= (i - jk) \bmod n. \end{aligned}$$

For each $i = 0, 1, \cdots, n-1$, we define a function $h_i : V(C_n) \to X$ by

$$h_i(v_j) = \begin{cases} \phi_i(v_j)z - \frac{x}{2}, & \text{if } I_{i,j} = \emptyset \text{ or } ||I_{i,j}|| \geqslant \frac{x}{2}; \\ \\ a_j, & \text{if } I_{i,j} \neq \emptyset,\ ||I_{i,j}|| < \frac{x}{2} \text{ and } a_j \in X_{\phi_i(v_j)}; \\ \\ a_j + 2, & \text{if } I_{i,j} \neq \emptyset,\ ||I_{i,j}|| < \frac{x}{2} \text{ and } a_j + 2 \in X_{\phi_i(v_j)}, \end{cases}$$

where $I_{i,j} = X_{\phi_i(v_j)} \cap L(v_j)$ and $||I_{i,j}||$ is the length of $I_{i,j}$.

Analogously, we define functions g_i for $0 \leqslant i \leqslant n-1$, by replacing $\phi_i(v_j)$ by $\psi_i(v_j)$ in the above definition of h_i.

It suffices to show that h_i or g_i for some i is a circular L-colouring for C_n. Note, the function h_i or g_i becomes *invalid* as a circular L-colouring only if at least one of the following occurs:

(a) $I_{i,j} = \emptyset$ for some j (that is, $m(v_j) = \phi_i(v_j)$ or $m(v_j) = \psi_i(v_j)$).

(b) $|h_i(v_j) - h_i(v_{j+1})|_r < 1$ or $|g_i(v_j) - g_i(v_{j+1})|_r < 1$ for some j.

To deal with the situations in (a) and (b), we define two types of edges on C_n, namely, saving and wasting edges. A directed edge $(v_t, v_{t'})$, where $t, t' = 0, 1, \cdots, n-1$ and $t' = t \pm 1$, is called *tight* if $a_t \in X_j$, $a_{t'} + 2 \in X_{j+k}$ for some j, and $\alpha(v_t) + \beta(v_{t'}) < \frac{x}{2}$. An edge $v_t v_{t'}$, $t' = t \pm 1$, on C_n is called

saving if $|m(v_t) - m(v_{t'})|_n = k$, and

wasting if $(v_t, v_{t'})$ or $(v_{t'}, v_t)$ is tight.

The sets of saving and wasting edges are denoted by S and W, respectively. Note, $S \cap W = \emptyset$.

Consider (a). For each $j = 1, 2, \cdots, n-1$, $\overline{L}(v_j)$ contains all except at most one number from 0 to $n-1$, so v_j would result in at most two functions, say h_i and $g_{i'}$, invalid. Precisely, this occurs when $m(v_j) = \phi_i(v_j) = \psi_{i'}(v_j)$. On the other hand, for each saving edge, say $v_t v_{t'} \in S$, v_t and $v_{t'}$ together would only make at most three functions invalid, instead of 4 by considering v_t and $v_{t'}$ separately, because if $m(v_t) = \phi_i(v_t) = \psi_{i'}(v_t)$ then $m(v_{t'}) = m(v_t) \pm k$ which is equal to either $\phi_i(v_{t'})$ or $\psi_{i'}(v_{t'})$, as $n = 2k+1$. Hence, by considering vertices v_i, $i = 1, 2, \cdots, n-1$, and edges in S, we conclude at most $2n - 2 - |S|$ functions of h_i's and g_i's are invalid.

Now consider (b). Notice that since $kz - \frac{x}{2} = 1$, by definitions of h_i and g_i, (b) occurs only when $v_j v_{j'}$ is a wasting edge. As each wasting edge results in at most one more function invalid, together with (a) at most $2n - 2 - |S| + |W|$ functions are invalid. As there are $2n$ functions to choose, we conclude that if $|S| \geqslant |W|-1$ then there exists some i such that h_i or g_i is a circular L-colouring for C_n.

It remains to consider the case that $|S| < |W|-1$. We define a *new* scaling on $S(r)$ by rotating the previously *old* scaling clockwise (that is, in the increasing order on $S(r)$) by $\frac{x}{2}$. Notice, the position of $L(v)$ on $S(r)$ for each v remains the same. Analogously, we define the corresponding parameters and functions for the new scaling the same as in the old scaling, and denote them accordingly by X_i^*, a_i^*, $m^*(v)$, $\alpha^*(v)$, $\beta^*(v)$, h_i^*, g_i^*, W^*, S^*, etc. For instance, we now have $L(v_0) = [\frac{x}{2}, 2+\frac{x}{2}]_r$ (that is, $a_0^* = \frac{x}{2}$).

Claim. $W \subseteq S^*$ and $W^* \subseteq S$.

Proof. To prove $W \subseteq S^*$, let $(v_t, v_{t'})$ be a tight edge in the old scaling. Then $a_t \in X_j$, $a_{t'}+2 \in X_{j+k}$ for some j, and $\alpha(v_t)+\beta(v_{t'}) < \frac{x}{2}$. In the new scaling, $\frac{x}{2} \leqslant a_t^* < x$ and $L(v_{t'}) \cap X_{j+k}^* = \emptyset$. Hence, we get $m^*(v_t) = j-1$ and $m^*(v_{t'}) = j+k$, implying $|m^*(v_t) - m^*(v_{t'})|_n = k$, so $v_t v_{t'} \in S^*$.

Similarly one can prove that $W^* \subseteq S$. □

As before, among the $2n$ new functions h_i^*'s and g_i^*'s from the new scaling, at most $2n - |S^*| + |W^*|$ are invalid as circular L-colourings for C_n. Because $|S| < |W|-1$, by the Claim, we get $2n - |S^*| + |W^*| \leqslant 2n-2$. This implies that there exists some h_i^* or g_i^* which is a circular L-colouring for C_n. The proof of Lemma 18 is complete. □

5 Generalized theta graphs

As observed earlier, $K_{2,n}$ is circular consecutive 2-choosable. Indeed, by definition $K_{2,n}$ is the generalized theta graph $\theta_{\underbrace{2,2,\cdots,2}_{n}}$. Hence, to study Question 1 it is natural to consider the following:

Question 2. *For which positive integers $k_1, k_2, \cdots, k_n$, the generalized theta graph $\theta_{k_1,k_2,\cdots,k_n}$ is circular consecutive 2-choosable?*

In this section, we provide partial answers to Question 2. First we consider circular L-colourings of the graph $\theta_{2,2,2}$ for some special colour-list assignment L.

Let the three paths of length 2 in $\theta_{2,2,2}$ be (x, z_1, x'), (x, z_2, x') and (x, z_3, x'). Assume $0 < \epsilon \leqslant 1/2$. Let $0 < \delta \leqslant (1-\epsilon)/3$ and $r = 4-\epsilon$. Let

$l : V(\theta_{2,2,2}) \to [0, 4-\epsilon)$ be defined as

$$l(v) = \begin{cases} 0, & \text{if } v = x, \\ 2+2\delta, & \text{if } v = x', \\ r-1-\delta, & \text{if } v = z_1, \\ r-1+3\delta+\epsilon, & \text{if } v = z_2, \\ 1+\delta, & \text{if } v = z_3, \end{cases}$$

Lemma 19. *Let $l : V(\theta_{2,2,2}) \to [0, 4-\epsilon)$ be defined as above. Let $L(v) = (l(v), l(v)+2+\delta)_r$ for $v \in \theta_{2,2,2}$. If f is a circular L-colouring of $\theta_{2,2,2}$, then*

$$f(x) \in (0, 4\delta+\epsilon)_r \quad \text{and} \quad f(x') \in (r-\delta, 3\delta+\epsilon)_r.$$

Proof. Assume the lemma is not true and f is a circular L-colouring of $\theta_{2,2,2}$ for which $f(x) \notin (0, 4\delta+\epsilon)_r$ or $f(x') \notin (r-\delta, 3\delta+\epsilon)_r$.

First we consider the case that $f(x) \notin (0, 4\delta+\epsilon)_r$. Then $f(x) \in [4\delta+\epsilon, 2+\delta)_r$. (Refer to Figure 2 for the positions of the intervals $L(x), L(x'), L(z_1)$, $L(z_2)$ and $L(z_3)$.)

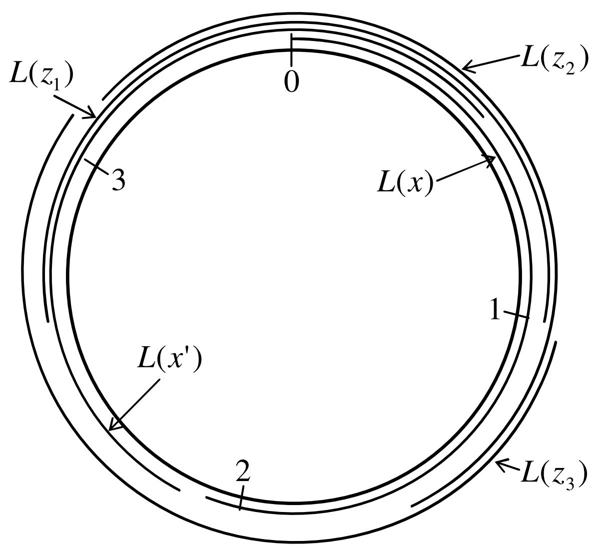

Figure 2: Locations of the intervals $L(x), L(x'), L(z_1), L(z_2), L(z_3)$, in the proof of Lemma 19

Since $L(z_2) = (r-1+3\delta+\epsilon, 1+4\delta+\epsilon)_r$, this forces $f(z_2) \in (r-1+3\delta+\epsilon, f(x)-1]_r$. As $L(x') = (2+2\delta, 3\delta+\epsilon)_r$, we must have $f(x') \in (2+2\delta, f(z_2)-1]_r$. On the other hand, we have $f(z_3) \in [f(x)+1, f(x')-1]_r$.

The four colours $f(x), f(z_2), f(x'), f(z_3)$ occur in the circle $S(r)$ in this cyclic order, and every two consecutive colours have distance at least 1. This is a contradiction, because $S(r)$ has length $r = 4 - \epsilon < 4$.

If $f(x') \notin (r-\delta, 3\delta+\epsilon)_r$, then $f(x') \in (2+2\delta, r-\delta]_r$. This forces $f(z_1) \in [f(x')+1, 1)_r$, which in turn forces $f(x) \in [f(z_1)+1, 2+\delta)_r$. As $f(z_3) \in [f(x)+1, f(x')-1]_r$ the four colours $f(x'), f(z_1), f(x), f(z_3)$ occur in $S(r)$ in this cyclic order, and every two consecutive colours have distance at least 1, which leads to the same contradiction. □

In the following, we use Lemma 19 to prove that $\theta_{2,2,2,n}$ has circular consecutive choosability greater than 2, provided that $n \neq 2, 4, 6$, and $\theta_{2,2,2,2,n}$ has circular consecutive choosability greater than 2 if $n \neq 2, 6$.

Theorem 20. *Suppose $n \geqslant 0$ is an integer. Then*

(1) $ch_{cc}(\theta_{2,2,2,2n+1}) \geqslant 2 + 1/(n+5)$.

(2) $ch_{cc}(\theta_{2,2,2,2n+8}) \geqslant 2 + 2/(4n+21)$.

Proof. Let the graph $\theta_{2,2,2,k}$ be obtained from the graph $\theta_{2,2,2}$, with vertices labeled as in Lemma 19, by adding the path $(x, y_1, y_2, \cdots, y_{k-1}, x')$.

First we show that $ch_{cc}(\theta_{2,2,2,2n+1}) \geqslant 2 + 1/(n+5)$ for any $n \geqslant 0$. It suffices to show that for any $0 < \epsilon \leqslant 1/2$, for $r = 4 - \epsilon$ and for $\delta = (1-\epsilon)/(n+5)$, there is a list assignment L which assigns to each vertex v an open interval of length $2+\delta$ of $S(r)$, for which there is no circular L-colouring of $\theta_{2,2,2,2n+1}$.

Let $l : V(\theta_{2,2,2,2n+1}) \to [0, 4-\epsilon)$ be defined so that the restriction to $\theta_{2,2,2}$ is the same as in Lemma 19, and

$$l(y_j) = \begin{cases} r + (4+t)\delta + \epsilon - 1, & \text{if } j = 2t+1, \\ r - t\delta, & \text{if } j = 2t. \end{cases}$$

We shall show that there is no circular L-colouring of $\theta_{2,2,2,2n+1}$. Assume to the contrary that there is a circular L-colouring f of $\theta_{2,2,2,2n+1}$. By Lemma 19, $f(x) \in (0, 4\delta+\epsilon)_r$ and $f(x') \in (r-\delta, 3\delta+\epsilon)_r$.

Since $L(y_1) = (r + 4\delta + \epsilon - 1, 1 + 5\delta + \epsilon)_r$ and $|f(x) - f(y_1)|_r \geqslant 1$, we conclude that $f(y_1) \in (1, 1+5\delta+\epsilon)_r$. Since $L(y_2) = (r-\delta, 2)_r$ and $|f(y_1) - f(y_2)|_r \geqslant 1$, we have $f(y_2) \in (r-\delta, 5\delta+\epsilon)_r$. Inductively, one can show that

$$\begin{aligned} f(y_{2j+1}) &\in (1 - j\delta, 1 + (j+5)\delta + \epsilon)_r \\ f(y_{2j}) &\in (r - j\delta, (j+4)\delta + \epsilon)_r. \end{aligned}$$

In particular, $f(y_{2n}) \in (r - n\delta, (n+4)\delta + \epsilon)_r$. As $f(x') \in (r - \delta, 3\delta + \epsilon)_r$ and $(n+5)\delta + \epsilon < 1$, we conclude that $|f(x') - f(y_{2n})|_r < 1$, in contrary to the assumption that f is a circular L-colouring of $\theta_{2,2,2,2n+1}$. This completes the proof of (1).

Next we prove that $ch_{cc}(\theta_{2,2,2,2n+8}) \geqslant 2 + 2/(4n+21)$ for any $n \geqslant 0$.

Let $\epsilon = \frac{2n+6}{4n+21}$, $r = 4 - \epsilon$ and $\delta = \frac{2}{4n+21}$. Let $l : V(\theta_{2,2,2,2n+8}) \to [0, 4-\epsilon)$ be defined so that the restriction of l to $\theta_{2,2,2}$ is as defined in Lemma 19 and

$$l(y_j) = \begin{cases} j - 2 + (3+j)\delta + \epsilon, & \text{if } 1 \leqslant j \leqslant 7, \\ 6 + (7+t)\delta + \epsilon, & \text{if } j = 2t \geqslant 8, \\ 7 - (t-3)\delta, & \text{if } j = 2t+1 \geqslant 9. \end{cases}$$

Now we shall prove that $\theta_{2,2,2,2n+8}$ has no circular L-colouring. Assume to the contrary that f is a circular L-colouring of $\theta_{2,2,2,2n+8}$. By Lemma 19, $f(x) \in (0, 4\delta + \epsilon)_r$ and $f(x') \in (r - \delta, 3\delta + \epsilon)_r$.

Similarly as in the proof of (1), we can prove by induction that for $j = 1, 2, \cdots, 7$,

$$f(y_j) \in (j, j + (j+4)\delta + \epsilon)_r.$$

For $j = 2t \geqslant 8$,

$$f(y_j) \in (8 - (t-4)\delta, 8 + (t+8)\delta + \epsilon)_r.$$

For $j = 2t+1 \geqslant 9$,

$$f(y_j) \in (7 - (t-3)\delta, 7 + (t+8)\delta + \epsilon)_r.$$

In particular,

$$f(y_{2n+7}) \in (7 - n\delta, 7 + (n+11)\delta + \epsilon)_r.$$

However, it is straightforward to verify that for any $a \in (r - \delta, 3\delta + \epsilon)_r$, for any $b \in (7 - n\delta, 7 + (n+11)\delta + \epsilon)_r$, $|a - b|_r < 1$. This is in contrary to our assumption that f is a circular L-colouring of $\theta_{2,2,2,2n+8}$. This completes the proof of (2). □

We do not know whether $ch_{cc}(\theta_{2,2,2,2n}) > 2$ for $n = 2, 3$. The next lemma shows that $ch_{cc}(\theta_{2,2,2,2,4}) > 2$.

Theorem 21. *$ch_{cc}(\theta_{2,2,2,2,4}) \geqslant 2 + 1/8$.*

Proof. Similar to Lemma 19, we first consider circular L-colourings of $\theta_{2,2,2,2}$, which is obtained from the graph $\theta_{2,2,2}$ in Lemma 19 by adding the path (x, z_4, x'). Let $l : V(\theta_{2,2,2,2}) \to [0, 4 - \epsilon)$ be defined such that the restriction of l to $\theta_{2,2,2,2} \backslash \{z_4\}$ is the same as in Lemma 19, and let $l(z_4) = r - 1 + \delta + \epsilon/2$.

Claim 1. *If f is a circular L-colouring of $\theta_{2,2,2,2}$ then either*

$$f(x) \in (0, 2\delta + \epsilon/2)_r, \quad \text{and} \quad f(x') \in (-\delta, 2\delta + \epsilon/2)_r$$

or

$$f(x) \in (\delta + \epsilon/2, 4\delta + \epsilon)_r, \quad \text{and} \quad f(x') \in (\delta + \epsilon/2, 3\delta + \epsilon)_r.$$

Proof. If the claim is not true, then by using Lemma 19, we conclude that one of $f(x), f(x')$ lies in the interval $(-\delta, \delta + \epsilon/2]_r$ and the other lies in the interval $[2\delta + \epsilon/2, 4\delta + \epsilon)_r$. Since z_4 is adjacent to both x and x', there is no legal colour for z_4 in the interval $L(z_4)$. This proves the claim. $\square$

Let $l : V(\theta_{2,2,2,2,4}) \to [0, 4 - \epsilon)$ be defined so that the restriction of l to $\theta_{2,2,2,2}$ is as in Claim 1 and for $j = 1, 2, 3$, $l(y_j) = j - 2 + (3 + j)\delta + \epsilon$. We shall show that, for appropriate ϵ and δ, $\theta_{2,2,2,2,4}$ has no circular L-colouring. Assume to the contrary that f is a circular L-colouring of $\theta_{2,2,2,2,4}$.

Let $\epsilon = 1/2$ and let $\delta = 1/8$. By Claim 1, we have two cases.

Case 1

$$f(x) \in (0, 2\delta + \epsilon/2)_r, \quad \text{and} \quad f(x') \in (-\delta, 2\delta + \epsilon/2)_r.$$

By using the proof of Theorem 20, we can show that $f(y_3) \in (3, 3 + 7\delta + \epsilon)_r$. Since $\epsilon = 1/2$ and $\delta = 1/8$, straightforward calculation shows that for any $a \in (-\delta, 2\delta + \epsilon/2)_r$, for any $b \in (3, 3 + 7\delta + \epsilon)_r$, we have $|a - b|_r < 1$, in contrary to our assumption that f is a circular L-colouring of $\theta_{2,2,2,2,4}$.

Case 2

$$f(x) \in (\delta + \epsilon/2, 4\delta + \epsilon)_r, \quad \text{and} \quad f(x') \in (\delta + \epsilon/2, 3\delta + \epsilon)_r.$$

Observe that, in comparison with Case 1, the possible colour of $f(x)$ is "shifted to the right" by a distance of $\delta + \epsilon/2$. By using the argument as in the proof of Theorem 20, we can show that $f(y_3) \in (3+\delta+\epsilon/2, 3+7\delta+\epsilon)_r$. Again, straightforward calculation shows that for any $a \in (\delta+\epsilon/2, 3\delta+\epsilon)_r$, for any $b \in (3+\delta+\epsilon/2, 3+7\delta+\epsilon)_r$, we have $|a-b|_r < 1$, in contrary to our assumption that f is a circular L-colouring of $\theta_{2,2,2,2,4}$. $\square$

We have shown that many generalized theta graphs G have $ch_{cc}(G) > 2$. A complete characterization of circular consecutive 2-choosable generalized theta graphs remains open and is interesting for further research. The following is a weaker version of this problem and is also open:

Question 3. *Is it true that for any positive integer k, $\theta_{2,2,2k+1}$ is circular consecutive 2-choosable?*

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