Chromatic Number of Distance Graphs Generated by the Sets $\{2, 3, x, y\}$

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Abstract

Let D be a set of positive integers. The distance graph generated by D has all integers $\mathbb{Z}$ as the vertex set; two vertices are adjacent whenever their absolute difference falls in D. We completely determine the chromatic number for the distance graphs generated by the sets $D = \{2, 3, x, y\}$ for all values x and y. The methods we use include the density of sequences with missing differences and the parameter involved in the so called "lonely runner conjecture." Previous results on this problem include: For x and y being prime numbers, this problem was completely solved by Voigt and Walther [30]; and other results for special integers of x and y were obtained by Kemnitz and Kolberg [18] and by Voigt and Walther [31].

Keywords: Distance graphs, chromatic number, density of sequences with missing differences, lonely runner conjecture.

1 Introduction

The study of distance graphs was motivated by the plane coloring problem: What is the minimum number of colors needed to color all the points on the Euclidean plane such that points with unit distance apart get different colors? It is known that four colors is necessary and seven colors is sufficient [16, 27], however, the exact number remains unknown.

Restricting the plane coloring problem to real numbers $\Re$ has an easy answer – using two colors alternatively on unit half open intervals. However, the problem becomes complex when more restrictions are added instead of only restricting different colors assigned to points with unit distance apart.

Eggleton, Erdős and Skilton [12] introduced the notion of coloring the real line with a given forbidden distance set D. Let D be a subset of $\Re$. Denote $G(\Re, D)$ the graph with the vertex set $\Re$ and edges connecting x and y if $|x-y| \in D$. In [12], the chromatic number of $G(\Re, D)$ for various families of sets D was investigated, such as $D = [1, \delta]$ or $D = (0, \delta)$ for some real $\delta > 1$.

We study a problem considered in [12], namely, when D is a set of positive integers. By isomorphism of components in $G(\Re, D)$, this problem is equivalent to considering the subgraph induced by all integers $\mathbb{Z}$ as the vertex set. Denote such a subgraph by $G(\mathbb{Z}, D)$ or simply $G(D)$, and call it the *integral distance graph* (or simply, *distance graph*) generated by the *distance set* D.

The chromatic number of distance graphs, denoted by $\chi(D)$, for different families of distance sets D has been studied extensively (cf. [8, 9, 12, 13, 14, 15, 18, 19, 23, 29, 31, 30, 34, 36]). Besides chromatic number, the fractional chromatic number and circular chromatic number of distance graphs have also been studied in the past two decades (cf. [24]).

Note that if x is a common divisor of all the elements in D, then $G(D)$ consists of disjoint subgraphs each isomorphic to $G(D/x)$ where $D/x = \{d/x : d \in D\}$. This implies that $\chi(D) = \chi(D/x)$. Hence, we assume throughout the article that $\gcd(D) = 1$.

The chromatic number of distance graphs for 2-element distance sets D is either 2 or 3. Namely, if D contains only odd numbers, then $\chi(D) = 2$; otherwise $\chi(D) = 3$. The chromatic number of distance graphs for 3-element sets were studied by Eggleton et al. [12], Chen et al. [9], and Voigt [29], and was completely settled by Zhu [36].

Theorem 1.1 [36] *Let $D = \{a, b, c\}$ with $a < b < c$ and $gcd(a, b, c) = 1$. Then*

$$\chi(D) = \begin{cases} 2 & \textit{if } a, b, c \textit{ are odd;} \\ 4 & \textit{if } D = \{1, 2, 3m\} \textit{ or } c = a + b \textit{ and } b - a \not\equiv 0 \textit{ (mod 3);} \\ 3 & \textit{otherwise.} \end{cases}$$

For 4-element sets, it was proved independently by Kemnitz and Marangio [19, 20], and by Liu and Zhu [26] that $\chi(D) = 5$ if $D = \{1, 2, 3, 4m\}$ or $D = \{x, y, y - x, y + x\}$ where x and y are odd integers. Moreover, Barajas and Serra [1] showed that, except these two families, the chromatic number for all other 4-element sets D is at most 4.

Theorem 1.2 [1] *Let D be a set of four positive integers. Then $\chi(D) \leqslant 4$ unless $D = \{1, 2, 3, 4m\}$ or $D = \{x, y, y - x, y + x\}$ where x and y are odd integers.*

The family $D = \{2, 3, x, y\}$ is of special interest in the study of *prime distance graphs*. Let $\mathbb{P}$ denote the set of all primes, and let D be a prime set, i.e., $D \subseteq \mathbb{P}$. As D does not contain any multiples of 4, the modular 4 function on integers gives a proper coloring for $G(D)$. Indeed, it was proved by Eggleton et al. [15] that $\chi(G(\mathbb{Z}, \mathbb{P})) = 4$, and for 3-element prime sets D, $\chi(D) = 4$ if and only if $D = \{2, 3, 5\}$.

Let D be a prime set. If $2 \notin D$, then $\chi(D) = 2$; if $3 \notin D$, then $\chi(D) \leqslant 3$, and the equality holds if and only if $2 \in D$ and $|D| \geqslant 2$. Hence, to study the chromatic number of a distance graph generated by a prime set it bounds to finding the chromatic number of D with $\{2, 3\} \subseteq D$, for which $\chi(D)$ is either 3 (D is called *Class 3*) or 4 (*Class 4*). It was proved in [15] that if $D \subseteq \mathbb{P}$ and $D = \{2, 3, x, x + 2\}$ (that is, x and $x + 2$ are twin primes), then D is Class 4. Furthermore, Voigt and Walther [30] completely characterized all 4-element Class 4 prime sets as follows.

Theorem 1.3 [30] *Let $D = \{2, 3, p, q\}$ be a set of primes with $p \geqslant 7$ and $q > p + 2$. Then $\chi(D) = 4$ holds if and only if (p, q) is one of the following:*

$$(11, 19), (11, 23), (11, 37), (11, 41), (17, 29), (23, 31), (23, 41), (29, 37).$$

For the sets $D = \{2, 3, x, x + s\}$ with general values $x, s \in \mathbb{Z}^+$, Kemnitz and Kolberg [18] determined the chromatic number for all $s \leqslant 9$. Voigt and Walther [31] proved that $\chi(\{2, 3, x, x+s\}) = 3$, if $s \geqslant 10$ and $x \geqslant s^2 - 6s + 3$.

Our aim is to completely determine the chromatic number of distance graphs generated by the sets $D = \{2, 3, x, y\}$ for all positive integers x and y. We utilize the close connections among coloring parameters (chromatic number, fractional chromatic number, and circular chromatic number) of distance graphs and two number theory problems, namely, the density of sets of forbidden differences, denoted by $\mu(D)$, and the parameter $\kappa(D)$ involved in the lonely runner conjecture.

In the next section, we introduce $\mu(D)$ and $\kappa(D)$, together with their useful relations with coloring parameters of distance graphs. In Section 3, we establish the main results, followed by a concluding remark in the last section.

2 $\mu(D)$ and $\kappa(D)$

Let S be a sequence of non-negative integers. For a non-negative integer n, let $S[n]$ denote the number of elements in S that are less than or equal to n. That is, $S[n] = |S \cap \{0, 1, 2, \ldots, n\}|$. The *upper density* and the *lower density* of S are defined respectively as the limit sup and limit inf of the ratio $S[n]/(n + 1)$. If the upper density and the lower density are equal, then the common value is called the *density* of S, and is denoted by $\sigma(S)$. That is,

$$\sigma(S) = \lim_{n\to\infty} S[n]/(n + 1).$$

Let D be a set of positive integers. A sequence S is called a *D-sequence* if $a-b \notin D$ for every $a, b \in S$. The density of sequences with missing differences in D, denoted by $\mu(D)$, is defined by:

$$\mu(D) = \sup \{\sigma(S) : \text{ S is a } D\text{-sequence}\}.$$

The parameter $\mu(D)$ is closely related to the fractional chromatic number of distance graphs. The *fractional chromatic number* of a graph G, denoted by $\chi_f(G)$, is the minimum ratio m/n $(m, n \in Z^+)$ of an (m/n)-coloring, where an (m/n)-coloring is a function on $V(G)$ to n-element subsets of $[m] =$

$\{1, 2, \cdots, m\}$ such that if $uv \in E(G)$ then $f(u) \cap f(v) = \emptyset$. There are several equivalent definitions of the fractional chromatic number; we refer the readers to the book by Scheinerman and Ullman [28].

Chang et al. [8] proved the following connection between distance graphs and $\mu(D)$: For any set of positive integers D,

$$\chi_f(G(\mathbb{Z}, D)) = 1/\mu(D). \tag{1.1}$$

Now we introduce the parameter $\kappa(D)$. For any real number x, let $||x||$ denote the minimum distance from x to an integer, that is, $||x|| = \min\{\lceil x \rceil - x, x - \lfloor x \rfloor\}$. For a given set D and any real t, denote $||tD||$ the smallest value $||td||$ among all $d \in D$. The kappa value of D, denoted by $\kappa(D)$, is the supremum of $||tD||$ among all real t. That is,

$$\kappa(D) = \sup\{\alpha \in (0, 1/2) : ||td|| \geqslant \alpha \text{ for some } t \in (0, 1), \text{ for all } d \in D\}.$$

The parameter $\kappa(D)$ is involved in the so called "lonely runner conjecture," which was first introduced by Wills [32]; the poetic name was due to Goddyn [4]. Suppose there are k runners running on a circular field with circumference l. Each runner is running at a constant speed, and all runners have different speeds. The conjecture asserts that for each runner, there exists some time t such that he becomes *lonely*, which means that all other runners are at least l/k apart from him. By taking ratio and letting $0 < t < 1$, we may assume without loss of generality that $l = 1$ and the speeds of the runners are integers. If we fix one runner r_i, and take absolute relative speeds of other runners to r_i, then we get a set with $k-1$ positive integers (of relative speeds). Hence, the conjecture is equivalent to: For any positive integral set D, it holds that $\kappa(D) \geqslant \frac{1}{|D|+1}$. The conjecture has been confirmed for $|D| \leqslant 6$ (that is, up to seven runners) [2, 3, 5, 10, 11], and remains open for $|D| \geqslant 7$.

Notice that for any set D, it holds that $\kappa(D) \leqslant \mu(D)$. The following two results will be frequently used to prove our main results in this article. The first was proved by Haralambis [17].

Lemma 2.1 [17] *Let D be a set of positive integers, and let $\alpha \in (0, 1]$. If for every D-sequence S with $0 \in S$ there exists a positive integer n such that $S[n]/(n+1) \leqslant \alpha$, then $\mu(D) \leqslant \alpha$.*

It is known [35] that $\chi_f(G) \leqslant \chi(G)$ holds for all graphs G, and $\chi(D) \leqslant \lceil 1/\kappa(D) \rceil$ holds for all sets D. Combining these with (1.1) we have:

Lemma 2.2 [8, 35] *For any given distance set D, it holds that*

$$1/\mu(D) \leqslant \chi(D) \leqslant \lceil 1/\kappa(D) \rceil.$$

Corollary 2.3 *Let D be a set of positive integers. If $\kappa(D) \geqslant 1/3$, then $\chi(D) \leqslant 3$; if $\mu(D) < 1/3$, then $\chi(D) \geqslant 4$.*

3 Main Results

Let $D = \{2, 3, x, y\}$, $x < y$. As we have learned in Section 1, $\chi(D)$ is either 3 or 4, except $\chi(\{1, 2, 3, 4m\}) = \chi(\{2, 3, 5, 8\}) = 5$. If $x = 1$, then $\chi(D) = 4$, except $\chi(D) = 5$ when $y = 4m$ for all positive integer m. Hence, in the rest of the article we assume $4 \leqslant x < y$.

For a given D-sequence S, we shall write elements of S in an increasing order, $S = s_0, s_1, s_2, \ldots$ with $s_0 < s_1 < s_2 < \ldots$, and denote its *difference sequence* by $\Delta(S) = \delta_0, \delta_1, \delta_2, \ldots$ where $\delta_i = s_{i+1} - s_i$. We call a subsequence of consecutive terms in S, $s_a, s_{a+1}, \ldots, s_{a+b-1}$, generate a periodic interval of k copies, $k \geqslant 1$, if $s_{j(a+b)+i} = s_{a+i}$ for all $0 \leqslant i \leqslant b-1$, $1 \leqslant j \leqslant k-1$. We denote such a periodic subsequence of S by $(s_a, s_{a+1}, \ldots, s_{a+b-1})^k$. If the periodic interval repeats infinitely we simply denote it by $(s_a, s_{a+1}, \ldots, s_{a+b-1})$. A sequence can be a mixture of some finite periodic intervals with an infinite periodic subsequence. For instance, we denote $S = a, b, c, a, b, c, d, d, d, e, f, e, f, e, f, e, f, \ldots$ by $(a, b, c)^2(d)^3(e, f)$.

Observation 1 *A sequence of non-negative integers S is a D-sequence if and only if the following holds for any $a \leqslant b$:*

$$\sum_{i=a}^{b} \delta_i \notin D.$$

In the following two results, we use the parameter $\mu(D)$ to prove $\chi(D) = 4$ for some sets D.

Theorem 3.1 *Let $D = \{2, 3, 6, x\}$, $x \geqslant 4$. Then*

$$\chi(\mathbb{Z}, D) = \begin{cases} 4, & \textit{if } x \equiv 0, \pm 1, \pm 4 \pmod 9; \\ 3, & \textit{otherwise.} \end{cases}$$

Proof. Let $D = \{2, 3, 6, x\}$, $x \geqslant 4$. Assume $x \not\equiv 0, \pm 1, \pm 4 \pmod 9$. Define f by

$$f(z) = \begin{cases} 1, & \text{if } z \equiv 0, 1, 5 \pmod 9; \\ 2, & \text{if } z \equiv 3, 4, 8 \pmod 9; \\ 3, & \text{if } z \equiv 2, 6, 7 \pmod 9. \end{cases}$$

It is easy to check that f is a proper 3-coloring for $G(\mathbb{Z}, D)$. Hence the result follows for $x \not\equiv 0, \pm 1, \pm 4 \pmod 9$.

To complete the proof, by Corollary 2.3, it is enough to show that if $x \equiv 0, \pm 1, \pm 4 \pmod 9$, then $\mu(D) < 1/3$. Assume to the contrary, $\mu(D) \geqslant 1/3$. Then $\mu(D) > (1/3) - \epsilon$ for any $\epsilon > 0$. By Lemma 2.1, there exists a D-sequence S such that for any $n \geqslant 0$, $\frac{S[n]}{n+1} > (1/3) - \epsilon$. Hence, for any positive integer n, there exists a D-sequence S such that $S[3t] \geqslant t + 1$ holds for every $3t \leqslant n$. By taking n as large as needed, this implies, for instance, $S[0] \geqslant 1$, so $0 \in S$; $S[3] \geqslant 2$, so $1 \in S$ (as $2, 3 \in D$); $S[6] \geqslant 3$, so $5 \in S$; and $S[9] \geqslant 4$, so $9 \in S$ (since $6, 7, 8 \notin S$). Continuing this process, we conclude that S has the infinite periodic difference sequence $\Delta(S) = (1, 4, 4)$, and so D does not contain any number from $9\mathbb{Z}^* \cup (9\mathbb{Z}^* \pm 1) \cup (9\mathbb{Z}^* \pm 4)$, contradicting $x \equiv 0, \pm 1, \pm 4 \pmod 9$. ∎

Theorem 3.2 *Let $D = \{2, 3, 10, x\}$ or $D = \{2, 3, 4, x\}$, $x \geqslant 5$. Then*

$$\chi(\mathbb{Z}, D) = \begin{cases} 4, & \textit{if } x \equiv 0, \pm 1 \pmod 6; \\ 3, & \textit{otherwise.} \end{cases}$$

Proof. Let $D = \{2, 3, 10, x\}$ or $D = \{2, 3, 4, x\}$, $x \geqslant 5$. Assume $x \not\equiv 0, \pm 1 \pmod 6$. Define f by

$$f(z) = \begin{cases} 1, & \text{if } z \equiv 0, 1 \pmod 6; \\ 2, & \text{if } z \equiv 2, 3 \pmod 6; \\ 3, & \text{if } z \equiv 4, 5 \pmod 6. \end{cases}$$

It is easy to check that f is a proper 3-coloring for $G(\mathbb{Z}, D)$.

To complete the proof, by Corollary 2.3, it is enough to show that if $x \equiv 0, \pm 1 \pmod 6$ then $\mu(D) < 1/3$. Assume to the contrary, $\mu(D) \geqslant 1/3$. Then $\mu(D) > (1/3) - \epsilon$ for any $\epsilon > 0$. Similar to the proof of Theorem 3.1, by Lemma 2.1, there exists a D-sequence S such that $S[3t] \geqslant t+1$ holds for every t with $3t \leqslant n$, where n is a positive integer as large as needed.

Now consider $D = \{2, 3, 10, x\}$. As $2, 3, 10 \in D$, it must be either $\{0, 1, 5, 6\} \subseteq S$ or $\{0, 1, 6, 7\} \subseteq S$. In either case, by considering the values of t with $S[3t] \geqslant t+1$, we conclude that $\Delta(S)$ must be one of the following:

$$(1,5), \quad (1,4,1,6), \text{ or } (1,4,1,6)^k(1,5) \text{ for some } k \geqslant 1.$$

For any of the above, D does not contain any number from $6\mathbb{Z}^* \cup (6\mathbb{Z}^* \pm 1)$. Hence, $x \not\equiv 0, \pm 1 \pmod 6$. This completes the proof for $D = \{2, 3, 10, x\}$.

For $D = \{2, 3, 4, x\}$, by a similar argument we get $\Delta(S) = (1, 5)$, so $x \not\equiv 0, \pm 1 \pmod 6$. ∎

A discrete version of $\kappa(D)$ for finite sets D is given as follows. For a set of D, let $\lambda, p \leqslant 2 \max D$ be positive integers, $\gcd(\lambda, p) = 1$. Denote

$$|\lambda D|_p = \min\{|\lambda d|_p : d \in D\},$$

where $|a|_p = a \bmod p$. Then we have

Proposition 3.3 *For a finite set D of positive integers,*

$$\kappa(D) = \max\,\{(|\lambda D|_p/p : \;\; 1 \leqslant p, \;\; \lambda \leqslant 2\max D, \gcd(\lambda, p) = 1\}.$$

In the next result, we use the above definition of $\kappa(D)$ to show $\chi(D) = 3$ for some sets D.

Theorem 3.4 *Let $D = \{2, 3, x, x+10\}$, $x \geqslant 4$. Then*

$$\chi(\mathbb{Z}, D) = \begin{cases} 4 & \text{if } x = 5; \\ 3 & \text{otherwise.} \end{cases}$$

Proof. The proofs for $x = 5$ and 6 follow by the known result that $\chi(\{2, 3, 5\}) = 4$ and by Theorem 3.1, respectively. For the remaining values of x, it suffices to show that $\kappa(D) \geqslant 1/3$. Consider three cases.

Case 1: $x = 3k$, for $k \geqslant 3$. Let $n = 2x + 10 = 6k + 10$. Then $\lceil n/3 \rceil = 2k + 4$. It is enough to find λ such that the following holds for all $d \in D$:

$$2k + 4 \leqslant |\lambda d|_n \leqslant 4k + 6. \tag{3.1}$$

Let $A = [k + 2, k + 2 + \lfloor k/3 \rfloor]$. Then $|2y|_n$ and $|3y|_n$ satisfy (3.1) for any $y \in A$.

Since x and $x + 10$ are the inverse of each other on $\mathbb{Z}_n$ as $n = 2x + 10$, it is enough to show that there exists some $\lambda \in A$ such that $|\lambda x|_n$ satisfies (3.1). As $k \geqslant 3$, we have $k + 2, k + 3 \in A$. Assume k is even. Then $(k + 3)x = (6k + 10)(k/2) + 4k \equiv 4k \pmod{n}$. So $|\lambda x|_n$ satisfies (3.1) with $\lambda = k + 3$.

Assume k is odd. Then $(k+2)x = (6k+10)((k-1)/2) + 4k + 5 \equiv 4k + 5 \pmod{n}$. Hence $|\lambda x|_n$ satisfies (3.1) with $\lambda = k + 2$.

Case 2: $x = 3k + 1$, for $k \geqslant 1$. Let $n = 2x + 10 = 6k + 12$. Then $n/3 = 2k + 4$. It is enough to find λ such that for all $d \in D$,

$$2k + 4 \leqslant |\lambda d|_n \leqslant 4k + 8. \tag{3.2}$$

Let

$$A = [k + 2, k + 2 + \lfloor (k + 2)/3 \rfloor].$$

Then $|2y|_n$ and $|3y|_n$ satisfy (3.2), for any $y \in A$.

As $k \geqslant 1$, we have $k + 2, k + 3 \in A$. Similar to Case 1, let $\lambda = k + 3$ or $k + 2$, when k is even or odd, respectively. One can easily verify that $|\lambda x|_n$ satisfies (3.2).

Case 3: $x = 3k + 2$, for $k \geqslant 2$. Let $n = 2x + 10 = 6k + 14$. Then $\lceil n/3 \rceil = 2k + 5$. It is enough to find λ such that

$$2k + 5 \leqslant |\lambda d|_n \leqslant 4k + 9. \tag{3.3}$$

Let

$$A = [k + 3, k + 3 + \lfloor k/3 \rfloor].$$

Then $|2y|_n$ and $|3y|_n$ satisfy (3.3), for any $y \in A$.

Similar to Case 1, letting $\lambda = k + 3$ or $k + 4$, when k is even or odd, respectively, it can be verified that $|\lambda x|_n$ satisfies (3.3). ∎

By all the above results we now only need to focus on the sets $D = \{2, 3, x, x + s\}$ with $x \geqslant 7$ and $s \geqslant 11$. Henceforth, we use an alternative ("continuous") definition of $\kappa(D)$.

Let $\alpha \in (0, 1/2)$. For a positive integer m, define

$$I_m(\alpha) = \{t \in (0, 1) : ||tm|| \geqslant \alpha\}.$$

That is,

$$I_m(\alpha) = \{t : q + \alpha \leqslant tm \leqslant q + 1 - \alpha, 0 \leqslant q \leqslant m - 1\}.$$

By the definition of $\kappa(D)$, we have

$$\kappa(D) = \sup\{\alpha \in (0, 1/2) : \cap_{i \in D} I_i(\alpha) \neq \emptyset\}.$$

Let $D = \{2, 3, x, y\}$. To show $\kappa(D) \geqslant 1/3$ it bounds to proving:

$$I_2(1/3) \cap I_3(1/3) \cap I_x(1/3) \cap I_y(1/3) \neq \emptyset.$$

In addition, because $I_2 \cap I_3 = [1/6, 2/9] \cup [7/9, 5/6]$, by symmetry, to show $\kappa(D) \geqslant 1/3$ it is enough to show:

$$[1/6, 2/9] \cap I_x(1/3) \cap I_y(1/3) \neq \emptyset.$$

For simplicity we denote $I_m(1/3)$ by I_m. Each I_m is the union of m disjoint intervals centered at $(2p + 1)/2m$, $0 \leqslant p \leqslant m - 1$, with width $1/3m$. Precisely, we write

$$I_{m,p} = [(3p + 1)/3m, (3p + 2)/3m],$$

where

$$I_m = \cup_{p=0}^{m-1} I_{m,p}.$$

We call each $I_{m,p}$ an I_m-*interval*. Notice that the gap between any two consecutive I_m-intervals, $I_{m,k}$ and $I_{m,k+1}$, is $2/(3m)$, twice the length of an I_m-interval.

Theorem 3.5 *Let $D = \{2, 3, x, x + s\}$ where $x = 7, 8$ and $s \geqslant 11$. Then $\chi(D) = 3$.*

Proof. Note, $I_7 \cap [1/6, 2/9] = [4/21, 2/9]$ and $I_8 \cap [1/6, 2/9] = [1/6, 5/24]$. It is easy to calculate that when $s \geqslant 11$, then $I_x \cap I_{x+s} \cap [1/6, 2/9] \neq \emptyset$ for $x = 7, 8$. Hence $\kappa(D) \geqslant 1/3$ and so $\chi(D) = 3$. ∎

Lemma 3.6 *For every $m \geqslant 12$, $m \neq 15, 16$, there exists some $I_{m,p} \subseteq [1/6, 2/9]$.*

Proof. Let $m \geqslant 12$ and $m \neq 15, 16$. It is straightforward to show that there exists some integer p, $0 \leqslant p \leqslant m-1$, such that $m \leqslant 6p+2$ and $9p+6 \leqslant 2m$. This implies, $(3p+1)/3m \geqslant 1/6$ and $(3p+2)/3m \leqslant 2/9$, so $I_{m,p} \subseteq [1/6, 2/9]$. ∎

Theorem 3.7 *Let $D = \{2, 3, x, y\}$, $12 \leqslant x < y$, $x \neq 15, 16$. Then $\chi(D) = 3$ for all $y \geqslant 2x$.*

Proof. By Lemma 3.6, there exists an interval $I_{x,p} \subseteq [1/6, 2/9]$. The length of $I_{x,p}$ is $1/(3x)$. Let $y \geqslant 2x$. The length of each I_y-interval is $1/(3y)$, and the gap between any two consecutive I_y-intervals is $2/(3y)$. Because $2/(3y) \leqslant 1/(3x)$ we conclude that $I_y \cap I_{x,p} \neq \emptyset$, implying $I_2 \cap I_3 \cap I_x \cap I_y \neq \emptyset$. So the result follows. ∎

Theorem 3.8 *Let $D = \{2, 3, x, y\}$. Then $\chi(D) = 3$ for all the following:*

(a) $x = 9$, $y \geqslant 36$;

(b) $x = 11$, $y \geqslant 66$;

(c) $x = 15$, $y \geqslant 60$;

(d) $x = 16$, $y \geqslant 48$.

Proof. Let $x = 9$. Then $I_9 \cap [1/6, 2/9] = [1/6, 5/27]$ with length $1/54$ (since $I_{9,1} = [4/27, 5/27]$). Let $y \geqslant 36$. Then the distance between any two consecutive I_y-interval's is $2/(3y) \leqslant 1/54$, implying $I_9 \cap I_y \cap [1/6, 2/9] \neq \emptyset$. Thus, $\chi(D) = 3$. Similarly, results for $x = 11, 15, 16$ can be obtained. ∎

Lemma 3.9 *If $m \geqslant 18k-1$, then k of the I_m-intervals are within $[1/6, 2/9]$.*

Proof. Write $m = 18k + i$, $i = -1, 0, 1, 2, \cdots, 16$. It is enough to show that there are k intervals within $[1/6, 2/9]$. Note, $I_{m,p} \subseteq [1/6, 2/9]$ if and only if $\frac{3p+1}{3m} \geqslant 1/6$ and $\frac{3p+2}{3m} \leqslant 2/9$, which implies

$$3k + \frac{i-2}{6} \leqslant p \leqslant 4k + \frac{2i-6}{9}.$$

It is easy to check that for $i = -1, 0, 1, \cdots, 16$, there are at least k values of p that would satisfy the above two inequalities. ∎

Theorem 3.10 *If $D = \{2, 3, x, x+s\}$ with $s \geqslant 11$ and $x \geqslant 53$, then $\chi(D) = 3$.*

Proof. By Corollary 2.3 it is enough to show $\kappa(D) \geqslant 1/3$ which is equivalent to show $I = I_x \cap I_{x+s} \cap [1/6, 2/9] \neq \emptyset$. Assume to the contrary, $I = \emptyset$. Suppose there are exactly k of the I_x-intervals in $[1/6, 2/9]$. By Lemma 3.9, $k \geqslant 3$. Let a and b be respectively the smallest and the largest values of $I_x \cap [1/6, 2/9]$. The length of the interval $[a, b]$ is $(k + 2(k-1))/3x = (3k-2)/3x$. Since $I = \emptyset$, so $I_{x+s} \cap I_x \cap [a, b] = \emptyset$. Hence, there are $k-1$ intervals of I_{x+s} within $[a, b]$, implying $(k-1+2k)/3(x+s) = (3k-1)/3(x+s) > (3k-2)/3x$. Therefore, $s < x/(3k-2)$. By the assumption that $s \geqslant 11$, we conclude $x \geqslant 33k - 21$. Since $k \geqslant 3$, by Lemma 3.9, there are $k+1$ intervals of I_x within $[1/6, 2/9]$, a contradiction. ∎

By Theorems 3.1, 3.2, 3.4, 3.8 and 3.10, to completely determine the value of $\chi(D)$ for $D = \{2, 3, x, y\}$, it remains to check the following:

- $x = 9$ and $20 \leqslant y < 36$;
- $x = 11$ and $22 \leqslant y < 66$;
- $x = 12, 13, 14$ or $17 \leqslant x \leqslant 52$, and $x + 11 \leqslant y < 2x$.
- $x = 15$ and $26 \leqslant y < 60$;
- $x = 16$ and $27 \leqslant y < 48$;

Because there are only a finite number of combinations of x and y values in the above, we established an algorithm to determine whether $\kappa(D) \geqslant 1/3$ by checking whether the interval I is an empty set or not for all the above sets D. We obtain:

Theorem 3.11 *Let $D = \{2, 3, x, x + s\}$ with $x \geqslant 9, x \neq 10$, and $s \geqslant 11$. Then $\kappa(D) \geqslant 1/3$ (so $\chi(D) = 3$), except $(x, x + s)$ falls in the following set:*

$$A = \{(9, 23), (11, 23), (11, 27), (11, 28), (11, 32), (11, 37), (11, 41), (11, 46),$$

$$(15, 35), (15, 41), (16, 37), (17, 29), (18, 31), (23, 36), (23, 41), (24, 37), (28, 41)\}.$$

For the sets D included in A in Theorem 3.11 we used an algorithm employing the idea presented in the proofs of Theorems 3.1 and 3.2 to determine the existence of a D-sequence S with $S[3t] \geqslant t + 1$ for all t; if S does not exist then $\mu(D) < 1/3$ and so $\chi(D) = 4$. It turned out that $\mu(D) < 1/3$ for all the sets A in Theorem 3.11, except $(x, x+s) \in \{(24, 37), (28, 41)\}$. In the following we show that the chromatic number for these two sets is also 4.

Notice that it can be easily proved that if f is a proper 3-coloring for $G(\mathbb{Z}, \{2, 3, x, y\})$ where $x \equiv 0, \pm 1 \pmod 6$ or $y \equiv 0, \pm 1 \pmod 6$, then there exist three consecutive integers $z, z + 1, z + 2$ that receive different colors, that is, $|\{f(z), f(z + 1), f(z + 2)\}| = 3$. For otherwise, a proper 3-coloring using colors a, b, c must be a periodic function on vertices by repeating the pattern a, a, b, b, c, c (period 6), contradicting the assumption that $x \equiv 0, \pm 1 \pmod 6$ or $y \equiv 0, \pm 1 \pmod 6$.

Theorem 3.12 *Let $D = \{2, 3, x, x + s\}$ with $x \geqslant 9$, $x \neq 10$, and $s \geqslant 11$. Then $\chi(D) = 4$ for every $(x, x + s)$ falls in A in Theorem 3.11.*

Proof. Let $(x, x + s) = (24, 37)$. Suppose to the contrary, $\chi(D) = 3$. Let f be a 3-coloring for $G(D)$. Without loss of generality, we assume $f(0) = a$, $f(1) = b$ and $f(2) = c$, implying $f(3) = c$, $f(4) = a$, $f(-1) = a$, $f(-2) = c$, $f(-4) = b$, and $f(6) = b$. Consider the following three cases.

Case 1: $f(32) = a$**.** Then we have the following:

$$\begin{aligned}
& f(35) = f(-5) = b, f(30) = f(8) = c \\
\rightarrow \quad & f(37) = c, f(-7) = a \\
\rightarrow \quad & f(34) = b \\
\rightarrow \quad & f(36) = c, f(10) = a \\
\rightarrow \quad & f(38) = a, f(12) = f(13) = b \\
\rightarrow \quad & f(40) = b, f(14) = c \\
\rightarrow \quad & f(16) = a, f(-10) = b \\
\rightarrow \quad & f(-8) = f(19) = c \\
\rightarrow \quad & f(43) = a \\
\rightarrow \quad & f(45) = b \\
\rightarrow \quad & f(47) = c \\
\rightarrow \quad & f(23) = b \\
\rightarrow \quad & f(26) = a \\
\rightarrow \quad & \text{impossible to color -11.}
\end{aligned}$$

Case 2: $f(32) = b$**.** We have

$$\begin{aligned}
& f(35) = a \\
\rightarrow \quad & f(33) = f(38) = c \\
\rightarrow \quad & f(30) = f(9) = a, f(36) = b \\
\rightarrow \quad & f(27) = b, f(-7) = f(12) = c \\
\rightarrow \quad & f(24) = c, f(-10) = a \\
\rightarrow \quad & f(14) = b \\
\rightarrow \quad & f(17) = a \\
\rightarrow \quad & f(20) = c, f(15) = b \\
\rightarrow \quad & f(23) = b, f(18) = a \\
\rightarrow \quad & \text{impossible to color 21.}
\end{aligned}$$

Case 3: $f(32) = c$**.** We have

$$\begin{aligned} & f(30) = f(8) = a \\ \to\ & f(33) = f(-7) = c, f(27) = f(5) = b \\ \to\ & f(36) = b, f(24) = c, f(-10) = f(9) = a \\ \to\ & f(39) = a, f(12) = c \\ \to\ & f(42) = c, f(-12) = f(15) = b \\ \to\ & f(-9) = a \\ \to\ & f(-6) = c \\ \to\ & f(-3) = b \\ \to\ & f(21) = a \\ \to\ & \text{impossible to color 18.} \end{aligned}$$

Therefore, $\chi(\{2, 3, 24, 37\}) = 4$.

Now consider $D = \{2, 3, 28, 41\}$. Suppose, to the contrary, $\chi(D) = 3$. Let f be a 3-coloring for $G(D)$. Without loss of generality, we assume $f(0) = a$, $f(1) = b$ and $f(2) = c$, implying $f(3) = c$, $f(4) = a$, $f(-1) = a$, $f(-2) = c$, $f(-4) = b$, and $f(6) = b$. Consider the following three cases.

Case 1: $f(36) = a$**.** Then we have the following:

$$\begin{aligned} & f(39) = f(-5) = b, f(8) = f(34) = c \\ \to\ & f(41) = c, f(11) = f(-7) = a \\ \to\ & f(13) = f(38) = b, f(9) = c \\ \to\ & f(10) = a \\ \to\ & f(12) = b \\ \to\ & f(40) = f(14) = c \\ \to\ & f(37) = f(42) = f(16) = a \\ \to\ & f(44) = b, f(35) = c \\ \to\ & f(33) = b, f(-6) = a \\ \to\ & f(5) = a, f(-8) = c \\ \to\ & f(46) = c \\ \to\ & f(18) = b \\ \to\ & f(21) = c \\ \to\ & f(23) = a \\ \to\ & \text{impossible to color 20.} \end{aligned}$$

Case 2: $f(36) = b$**.** Then we have

$$\begin{array}{rl} & f(39) = f(-5) = a \\ \to & f(42) = f(-7) = f(37) = c \\ \to & f(40) = b, f(34) = f(9) = a \\ \to & f(43) = a, f(12) = c, f(31) = b \\ \to & f(15) = b, f(-10) = a \\ \to & f(-13) = c \\ \to & \text{impossible to color 28.} \end{array}$$

Case 3: $f(36) = c$**.** Then we have

$$\begin{array}{rl} & f(34) = a \\ \to & f(37) = f(-7) = c, f(31) = b \\ \to & f(9) = f(-10) = a, f(40) = b \\ \to & f(12) = c, f(43) = a \\ \to & f(15) = b \\ \to & f(-13) = c \\ \to & \text{impossible to color 28.} \end{array}$$

Therefore, $\chi(\{2, 3, 28, 41\}) = 4$. ∎

4 Concluding Remark

The chromatic numbers of the distance graphs generated by $D = \{2, 3, x, y\}$ for all values of x and y are completely determined. Let $x = 1$. Then $\chi(\{1, 2, 3, y\}) = 5$ if $y = 4m$; otherwise $\chi(\{1, 2, 3, y\}) = 4$. For $x \geqslant 4$, $\chi(D)$ is either 3 or 4, except $\chi(\{2, 3, 5, 8\}) = 5$; the two tables below list all the sets D with chromatic number 4.

We prove $\chi(D) = 3$ by either giving a proper 3-coloring (Theorems 3.1, 3.2) or by showing $\kappa(D) \geqslant 1/3$ (most cases). We prove $\chi(D) = 4$ by either verifying $\mu(D) < 1/3$ (most cases) or by showing the non-existence of a proper 3-coloring (Theorem 3). It turned out that all the sets D with chromatic number 3 have $\kappa(D) \geqslant 1/3$; and all sets D with chromatic number 4 satisfy $\mu(D) < 1/3$, except $D = \{2, 3, 24, 37\}$ and $D = \{2, 3, 28, 41\}$ which have $\mu(D) \geqslant 1/3$ (and $\kappa(D) < 1/3$).

The exact values of $\kappa(D)$ and $\mu(D)$ (or equivalently $\chi_f(D)$) for sets $D = \{2, 3, x, y\}$ are worthy for future investigation. Another research direction one might take is to apply the methods used in this article to study other families of sets D, such as the open problem of characterizing prime sets into Classes 3 and 4.

Table 1: Sets $D = \{2, 3, x, x+s\}$ with $\chi(D) = 4$ for $1 \leqslant s \leqslant 10$.

s	**x**	**References**
1	4, 5, 10	[18]
2	$x \not\equiv 2 \pmod 6$	[18]
3	$x \not\equiv 3 \pmod 9$, $x \neq 5$	[18]
4	5, 6	[18]
5	5	[18]
6	5	[18]
7	4, 5, 6, 10, 11, 12, 16, 17, 22	[18]
8	4, 5, 6, 9, 10, 11, 13, 15, 18, 19, 23, 24, 28, 29, 33, 37, 42, 47	[18]
9	4, 5, 10	[18]
10	5	new

Table 2: Sets $D = \{2, 3, x, y\}$ with $\chi(D) = 4$ for $y \geqslant x + 11$.

x	**y**	**References**
4, 10	$y \equiv 0, \pm 1 \pmod 6$	new
5	all positive integers $y \neq 5$	[18, 20, 26]
6	$y \equiv 0, \pm 1, \pm 4 \pmod 9$	new
$x \geqslant 7, x \neq 10$	$(x, y) \in \{(9, 23), (11, 23), (11, 27), (11, 28), (11, 32),$ $(11, 37), (11, 41), (11, 46), (15, 35), (15, 41), (16, 37),$ $(17, 29), (18, 31), (23, 36), (23, 41), (24, 37), (28, 41)\}$	new (some in [30])

References

[1] J. Barajas and O. Serra, *Distance graphs with maximum chromatic number*, Disc. Math., 308 (2008) 1355 – 1365.

[2] J. Barajas and O. Serra, *The lonely runner with seven runners*, The Electronic Journal of Combinatorics, 15 (2008), #R48.

[3] U. Betke and J.M. Wills, *Untere Schranken für zwei dophantische approximations-Funktionen*, Monatsch. Math., 76 (1972) 214–217.

[4] W. Bienia, L. Goddyn, P. Gvozdjak, A. Sebő and M. Tarsi, *Flows, view obstructions, and the lonely runner*, J. Combin. Theory (B), 72 (1998) 1 – 9.

[5] T. Bohman, R. Holzman, D. Kleitman, *Six lonely runners*, Electronic J. Combinatorics, 8 (2001), Research Paper 3, 49 pp.

[6] D. G. Cantor and B. Gordon, *Sequences of integers with missing differences*, J. Combin. Th. (A), 14 (1973) 281 – 287.

[7] G. J. Chang, L. Huang and X. Zhu, *The circular chromatic numbers and the fractional chromatic numbers of distance graphs*, Europ. J. of Combin., 19 (1998) 423 – 431.

[8] G. Chang, D. Liu and X. Zhu, *Distance graphs and T-coloring*, J. Combin. Th. (B), 75 (1999) 159 – 169.

[9] J. Chen, G. J. Chang and K. Huang, *Integral distance graphs*, J. Graph Theory, 25 (1997) 287 – 294.

[10] T. W. Cusick, *View-obstruction problems in n-dimensional geometry*, J. Combin. Theory Ser. A, 16 (1974) 1 – 11.

[11] T. W. Cusick and C. Pomerance, *View-obstruction problems, III*, J. Number Th., 19 (1984), 131–139.

[12] R. B. Eggleton, P. Erdős and D. K. Skilton, *Colouring the real line*, J. Combin. Theory (B), 39 (1985) 86 – 100.

[13] R. B. Eggleton, P. Erdős and D. K. Skilton, *Research problem 77*, Disc. Math., 58 (1986) 323.

[14] R. B. Eggleton, P. Erdős and D. K. Skilton, *Update information on research problem 77*, Disc. Math., 69 (1988) 105.

[15] R. B. Eggleton, P. Erdős and D. K. Skilton, *Colouring prime distance graphs*, Graphs and Combinatorics, 6 (1990) 17 – 32.

[16] H. Hadwiger, H. Debrunner and V. Klee, *Combinatorial Geometry in the Plane*, Holt Rinehart and Winston, New York, 1964.

[17] N. M. Haralambis, *Sets of integers with missing differences*, J. Combin. Th. (A), 23 (1977) 22 – 33.

[18] A. Kemnitz and H. Kolberg, *Coloring of integer distance graphs*, Disc. Math., 191 (1998) 113 – 123.

[19] A. Kemnitz and M. Marangio, *Colorings and list colorings of integer distance graphs*, Congr. Numer., 151 (2001) 75 – 84.

[20] A. Kemnitz and M. Marangio, *Chromatic numbers of integer distance graphs*, Disc. Math., 233 (2001) 239–246.

[21] P. Lam and W. Lin, *Coloring distance graphs with intervals as distance sets*, European J. of Combin., 26 (2005) 1216 – 1229.

[22] W. Lin, P. Lam and Z. Song, *Circular chromatic numbers of some distance graphs*, Disc. Math., 292 (2005) 119 – 130.

[23] D. Liu, *T-coloring and chromatic number of distance graphs*, Ars Combin., 56 (2000) 65–80.

[24] D. Liu, *From Rainbow to the Lonely Runner: A survey on coloring parameters of distance graphs*, Taiwanese J. Math., 12 (2008) 851 – 871.

[25] D. Liu and X. Zhu, *Distance graphs with missing multiples in the distance sets*, J. Graph Theory, 30 (1999) 245 – 159.

[26] D. Liu and X. Zhu, *Fractional chromatic number and circular chromatic number for distance graphs with large clique size*, J. Graph Theory, 47 (2004) 129 – 146.

[27] L. Moser and W. Moser, *Solution to problem 10*, Canad. Math. Bull., 4 (1961) 187 – 189.

[28] E. R. Scheinerman and D. H. Ullman, *Fractional Graph Theory*, Wiley-Interscience Series in Discrete Mathematics and Optimization, 1997.

[29] M. Voigt, *Colouring of distance graphs*, Ars Combinatoria, 52 (1999) 3 – 12.

[30] M. Voigt and H. Walther, *Chromatic number of prime distance graphs*, Discrete Appl. Math., 51 (1994) 197 – 209.

[31] M. Voigt and H. Walther, *On the chromatic number of special distance graphs*, Disc. Math., 97 (1991) 395 – 397.

[32] J. M. Wills, *Zwei Sätze über inhomogene diophantische appromixation von irrationlzahlen*, Monatsch. Math., 71 (1967) 263 – 269.

[33] J. Wu and W. Lin, *Circular chromatic numbers and fractional chromatic numbers of distance graphs with distance sets missing an interval*, Ars Comb., 70 (2004) 161 – 168.

[34] X. Zhu, *Pattern periodic coloring of distance graphs*, J. Combin. Theory (B), 73 (1997) 195 – 206.

[35] X. Zhu, *Circular chromatic number: A survey*, Discrete Mathematics, 229 (2001) 371 – 410.

[36] X. Zhu, *The circular chromatic number of distance graphs with distance sets of cardinality* 3, J. Graph Theory, 41 (2002) 195 – 207.

[37] X. Zhu, *The circular chromatic number of a class of distance graphs*, Disc. Math., 265/1-3 (2003) 337 – 350.