Circular Distance Two Labeling and the λ-Number for Outerplanar Graphs

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Abstract

Let G be a graph. A circular distance two labeling with span k is a function $f : V(G) \to \{0, 1, 2, \cdots, k-1\}$ such that: 1) $2 \leq |f(u) - f(v)| \leq k-2$ if u and v are adjacent; and 2) $f(u) \neq f(v)$ if u and v are of distance two apart. We denote by $\lambda_c(G)$ the smallest span of a circular distance two labeling for G. Let $\Delta(G)$ be the maximum degree of G. We prove, for any outerplanar graph G, $\lambda_c(G) = \Delta(G) + 3$, provided $\Delta(G) \geq 15$; and $\lambda_c(G) \leq \Delta(G) + 4$, provided $\Delta(G) \geq 11$. It is also shown that there exist outerplanar graphs G with $\Delta(G) = 2, 3, 4, 5$ for which $\lambda_c(G) = \Delta(G) + 4$. Moreover, we prove that $\lambda_c(G) \leq \Delta(G) + 5$ for any triangulated outerplanar graph, and $\lambda_c(G) \leq \Delta(G) + 7$ for any outerplanar graph. Immediate consequences of our results include that $\lambda(G) \leq \Delta(G) + 2$ for any outerplanar graphs with $\Delta(G) \geq 15$, where $\lambda(G)$ is the minimum k of a k-$L(2,1)$-labeling (or distance two labeling) for G.

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1 Introduction

Distance two labeling (or $L(2,1)$-labeling) is motivated by the channel assignment problem (cf. [7]). The task is to assign one non-negative integral channel to each of the given transmitters or stations so that interference is avoided, and the span of all the channels used is minimized.

Suppose that we are dealing with two levels of interference – major and minor. Major interference occurs between two close transmitters. To avoid it, the difference of the channels assigned to such a pair of transmitters must be at least 2. Minor interference occurs between two transmitters that share a common close neighbor. To avoid it, the difference of the channels assigned to such a pair of transmitters must be at least 1.

Let $G = (V, E)$ be the graph where each vertex represents a transmitter, and two vertices are adjacent if the corresponding transmitters are close. The above channel assignment corresponds to an $L(2,1)$-*labeling* of G, which is defined to be a function $f : V(G) \to \{0, 1, 2, \cdots\}$ such that the following are satisfied, where $d_G(u, v)$ denotes the distance between u and v in G:

- $|f(x) - f(y)| \geq 2$, if $d_G(x, y) = 1$; and
- $|f(x) - f(y)| \geq 1$, if $d_G(x, y) = 2$.

The *span* of f is defined as $\text{span}(f) = \max_{x \in V} f(x) - \min_{x \in V} f(x)$. If $\text{span}(f) = k$, then f is called a k-$L(2,1)$-labeling. Without loss of generality, for convenience, we assume that $\min_{x \in V} f(x) = 0$, and hence $\max_{x \in V} f(x) = k = \text{span}(f)$. The numbers $0, 1, 2, \cdots, k$ are called *colors* (or labels). The λ-*number* of G, denoted by $\lambda(G)$, is the minimum k such that G admits a k-$L(2,1)$-labeling.

A *circular distance two labeling* with span k (or a k-$L_c(2,1)$-labeling) of a graph G is a function, $f : V(G) \to \{0, 1, 2, \cdots, k-1\}$, such that the following are satisfied:

- $|f(x) - f(y)|_k \geq 2$, if $d_G(x, y) = 1$; and
- $|f(x) - f(y)|_k \geq 1$, if $d_G(x, y) = 2$,

where $|x - y|_k$, the *modular k circular difference* between x, y, is defined as $|x - y|_k = \min\{|x - y|, k - |x - y|\}$. The *circular λ-number* of G, denoted by $\lambda_c(G)$, is the smallest k such that G admits a k-$L_c(2,1)$-labeling. Circular distance two labeling and the values of $\lambda_c(G)$ for different families of graphs have been studied in [10, 11, 12, 13].

By definition, every $(k+1)$-$L_c(2,1)$-labeling is a k-$L(2,1)$-labeling, and every k-$L(2,1)$-labeling is a $(k+2)$-$L_c(2,1)$-labeling. Therefore, we have

$$\lambda(G) + 1 \leq \lambda_c(G) \leq \lambda(G) + 2. \tag{1.1}$$

The colors in circular distance two labeling are *symmetric* in the following sense. Let f be a k-$L_c(2,1)$-labeling of G. Then, for any $i \in \{0, 1, 2, \cdots, k-1\}$, the function defined by $f^*(u) = f(u) - i \pmod{k}$ is also a k-$L_c(2,1)$-labeling for G. The colors in distance two labeling does not have this property. For instance, the star $K_{1,n}$ has $\lambda(G) = n + 1$, and any optimal $L(2,1)$-labeling must assign to the center vertex either 0 or $n + 1$. This kind of asymmetry in colors sometimes causes difficulties in discussion. In this article, we take advantage of the symmetry of colors in circular distance two labeling to explore the value of $\lambda_c(G)$, which, by (1.1) gives good bounds for the λ-number of G.

The circular λ-number of graphs is closely related to the circular chromatic number of edge weighted graphs, a notion introduced by Mohar [14]. An *edge weighted graph* with vertex set V is a pair $\vec{G} = (V, A)$, where $A : V \times V \to R^+ \cup \{0\}$ is a weight assignment for the directed edges of G. For every directed edge $(u, v) \in \vec{E}$, we write $a_{uv} = A(u, v)$. For a positive real number p, denote by $S_p \subset R^2$ the circle with perimeter p centered at the origin of R^2. For any $x, y \in S_p$, let $l(x, y)$ denote the length of the arc from x to y on S_p in the clockwise direction. A *circular p-coloring* of $\vec{G}$ is a function $c : V \to S_p$ such that $l(c(u), c(v)) \geq a_{uv}$ for every directed edge $(u, v) \in \vec{E}$. The *circular chromatic number* $\chi_c(\vec{G})$ of the graph $\vec{G} = (V, A)$ is the infimum of all real numbers p for which there exists a circular p-coloring of $\vec{G}$.

For any graph G (un-directed), we construct an edge weighted graph $\vec{G}^2 = (V, A)$ by: 1) For each $uv \in E(G)$, let $a_{uv} = a_{vu} = 2$; 2) if u' and v' are distance two apart in G, then we add two directed edges (u', v'), (v', u') to $\vec{E}$ with $a_{u'v'} = a_{v'u'} = 1$; and 3) $a_{uv} = 0$ for all other pairs (u, v). It is not hard to verify that the following holds for any graph G [12]:

$$\lambda_c(G) = \lceil \chi_c(\vec{G}^2) \rceil.$$

Determining $\lambda(G)$ is an NP-complete problem, even restricted to special classes of graphs, such as graphs with diameter 2 [6], planar graphs, bipartite graphs, chordal graph, or split graphs (cf. [1]). Research on the parameter $\lambda(G)$ has been concentrated on finding good upper bounds for $\lambda(G)$. Denote by $\Delta(G)$ the maximum degree of G, or Δ when G is clear in the context. It is easy to see that for any graph G, $\lambda(G) \geq \Delta + 1$ and $\lambda_c(G) \geq \Delta + 3$. It was shown in [6] that for any G, $\lambda(G) \leq \Delta^2 + 2\Delta$. This bound was improved to $\lambda(G) \leq \Delta^2 + \Delta$ by Chang and Kuo [2], where the proof actually shows that $\lambda_c(G) \leq \Delta^2 + \Delta + 1$. A still open conjecture [6] states that $\lambda(G) \leq \Delta^2$ for any graph G. For special classes of graphs, better upper bounds are known. Below we list some of the known results on $\lambda(G)$ for some families of graphs.

Graphs	$\lambda(G)$	**Reference**
Trees	$\Delta + 1$ or $\Delta + 2$	Chang and Kuo [2]
Chordal	$\leq \frac{1}{4}(\Delta + 3)^2$	Sakai [16]
Diameter two	$\leq \Delta^2$	Griggs and Yeh [6]
Planar	$\leq 2.5\Delta + 90$	Molloy and Salavatipour [15]
Outerplanar (OP)	$\leq \Delta + 8$	Bodlaender et al. [1]
Triangulated OP	$\leq \Delta + 6$	Bodlaender et al. [1]

A graph is *outerplanar* if it can be embedded in the plane in such a way that all the vertices lie on the infinite face. We call such an embedding *outerplane graph*. An outerplanar graph is *triangulated* if it can be drawn as an outerplane graph such that each finite face is a triangle. In searching for the λ-number of outerplanar graphs, Bodlaender et al. [1] proposed:

Conjecture 1 *For any outerplanar graph G, $\lambda(G) \leq \Delta + 2$.*

Observe that, for any graph G,

$$\lambda_c(G) \geq \Delta + 3.$$

The main result of this article is:

Theorem 1 *For any outerplanar graph G with $\Delta \geq 15$, $\lambda_c(G) = \Delta + 3$.*

By (1.1), an immediate consequence of Theorem 1 is the confirmation of Conjecture 1 for outerplanar graphs with large maximum degree.

Corollary 2 *For any outerplanar graph G with $\Delta \geq 15$, $\lambda(G) \leq \Delta + 2$.*

For outerplanar graphs with smaller maximum degree, we prove the following two results:

Theorem 3 *Suppose G is an outerplanar graph. Then $\lambda_c(G) \leq \Delta(G) + 7$. Moreover, if G is triangulated, then $\lambda_c(G) \leq \Delta(G) + 5$.*

Theorem 4 *If G is an outerplanar graph and $\Delta(G) \geq 11$, then $\lambda_c(G) \leq \Delta(G) + 4$.*

Combining Theorem 3 with (1.1), we are able to improve the bounds of the λ-number for outerplanar graphs obtained in [1].

Note that, the condition $\Delta(G) \geq 15$ in Theorem 1 cannot be simply removed. In the last section of this article, we demonstrate the existence of outerplanar graphs G with $\Delta(G) = 2, 3, 4, 5$ and $\lambda_c(G) = \Delta(G) + 4$.

2 Structure of outerplanar graphs

Let G be an outerplanar graph. Then G can be transformed into a triangulated outerplane graph G_T by adding some edges. We call G_T a *triangulation* of G. There may exist many triangulations of G, however, we denote by G_T an arbitrary but fixed triangulation.

Let G be a triangulated outerplane graph. We define a *level function* l on $V(G)$, by recursion, such that $l(u) \neq l(v)$ if $u \sim v$. **Initially:** Choose an edge $e = u_1u_2$ on the infinite face and let $l(u_1) = 1$, $l(u_2) = 2$; we call e the *root edge* and u_1, u_2 the *root vertices*. **Inductively:** Let $X = \{v \in V(G) : l(v) \text{ is defined}\}$. While $X \neq V(G)$, choose a triangle (u, v, w) such that $v, w \in X$ and $u \in V(G) - X$. Assume $l(v) > l(w)$ (since $v \sim w$, by inductive hypothesis, $l(v) \neq l(w)$). Let $l(u) = l(v) + 1$. The vertices w, v are called the *major parent* and the *minor parent* of u, and are denoted by $w = f(u)$ and $v = m(u)$, respectively. It is easy to verify that, at each step, the subgraph $G[X]$ of G induced by X is still a triangulated outerplane graph. This implies that if, at some step, $u \in V(G) - X$ is contained in a triangle (u, v, w) such that $v, w \in X$, then the triangle is unique. Hence, for any non-root vertex u, the functions $l(u)$, $m(u)$ and $f(u)$ are well-defined. The following lemma follows from the definitions.

Lemma 5 *If u is a non-root vertex, then $f(u) \in \{m(m(u)), f(m(u))\}$. If $u' \neq u$ are two non-root vertices, then $\{f(u), m(u)\} \neq \{f(u'), m(u')\}$.*

If v is a parent of u, then u is called a *child* of v. If v is the major (respectively, minor) parent of u, then u is called a major (respectively, minor) child of v. If $m(u) = m(u')$, then u and u' are *siblings.* Note that a vertex may have many children. However, the following lemma shows that each vertex has at most one sibling and at most two children of the same level.

Lemma 6 *Suppose G is a triangulated outerplanar graph with level function l. Let v be a vertex and i a positive integer. Then v has at most two children u with $l(u) - l(v) = i$. In particular, v has at most two minor children and at most one sibling.*

Proof. Let $W_i = \{x : x \text{ is a child of } v \text{ with } l(x) = l(v) + i\}$. We prove by induction on i that $|W_i| \leq 2$. If $u \in W_1$, then $m(u) = v$. If v is a root vertex, then it follows from the definition that $|W_1| \leq 1$. If v is a non-root vertex, by Lemma 5, $f(u) \in \{f(v), m(v)\}$. Hence, the parents of u are either $\{v, f(v)\}$ or $\{v, m(v)\}$. By Lemma 5, there is at most one vertex whose parents are

$\{v, f(v)\}$ and at most one vertex whose parents are $\{v, m(v)\}$. Therefore $|W_1| \leq 2$. Assume $|W_k| \leq 2$ for some $k \geq 1$. If $u \in W_{k+1}$, then $v = f(u)$ and $m(u) \in W_k$. Since $|W_k| \leq 2$, it follows from Lemma 5 that $|W_{k+1}| \leq 2$.

If u' and u are siblings, then u and u' are both minor children of $m(u) = m(u')$, of level $l(m(u)) + 1$. As $m(u)$ has at most two children with level $l(m(u)) + 1$, we conclude that each vertex has at most one sibling. ∎

If G is a non-triangulated outerplanar graph, then we define the level function l on a triangulation G_T of G, and view l as a function on G. Similarly, parents, children and siblings are defined according to l in the same manner. Note that, as G is non-triangulated, a vertex u may not be adjacent to its parents.

Next, we define a lexicographic ordering $\prec$ on $V(G)$ by:

- If $l(u) < l(u')$, then $u \prec u'$. If $l(u) = l(u)$ and $f(u) \prec f(u')$, then $u \prec u'$. If $l(u) = l(u')$ and $f(u) = f(u')$, then $u \prec u'$ if and only if $m(u) \prec m(u')$.

By Lemma 5, $\prec$ is a linear ordering on $V(G)$. Throughout this article, we write $V(G)$ as $V(G) = \{v_1, v_2, \cdots, v_n\}$, where $v_i \prec v_j$ if and only if $i < j$. In particular, $v_1 = u_1, v_2 = u_2$ are the two root vertices.

Let t be an integer, $1 \leq t \leq n$. Denote $V_t = \{v_1, v_2, \cdots, v_t\}$. Let $w \in V$. We denote the number of neighbors of w in V_t by $s[w, t]$, that is:

$$s[w, t] = |\{v_j : j \leq t, v_j \sim w\}|.$$

Observe that, for any vertex $w = v_b$ of an outerplanar graph G, if $f(w) = v_i$ and $m(w) = v_j$, then $i < j < b$, $s[w, i] \leq 1$, and $s[w, j] = s[w, b] \leq 2$. Moreover, we have:

Lemma 7 *Let G be an outerplanar graph and G_T a triangulation of G. Let $w \in V(G)$. Suppose $w \sim v_t$ in G_T.*

(1) If $s[w, t] \geq 5$, then $w = f(v_t)$.

(2) If $s[w, t] \geq 7$, then $f(m(v_t)) = w$.

(3) If $s[w,t] \geq 9$, then $f(m(m(v_t))) = w$.

(4) If $m(v_t) = v_{t'}$, then $s[w,t] - s[w,t'] \leq 2$.

(5) If $w = f(v_l) = f(v_t)$, and $v_l \sim v_t$ for some l, then $|s[w,t] - s[w,l]| \leq 2$.

Proof. The neighbors of w in G_T, in the ordering $\prec$, are: First, the parents of w; secondly, the minor children of w; and finally, the major children of w. By Lemma 6, w has only two parents and at most two minor children. Hence, if $s[w,t] \geq 5$, then v_t must be a major child of w, i.e., $w = f(v_t)$. So (1) holds.

The rest of the lemma can be proved similarly, and we omit the details. ∎

3 Proofs of Theorems 3 and 4

Suppose G is an outerplanar graph with vertex set $V = \{v_1, v_2, \cdots, v_n\}$, ordered as in Section 2. To prove Theorems 1, 3 and 4, it suffices to find a k-$L_c(2,1)$-labeling for G, by the corresponding desired value of k. We regard the colors $\{0, 1, 2, \cdots, k-1\}$ on a circular palette, and all calculations are taken modulo k. Let C be a proper subset of colors on this color palette. A *segment* of C is a maximal interval of consecutive colors of C, i.e., a set I of colors of the form $I = \{j, j+1, \cdots, l\}$ such that $I \subset C$ and $j-1, l+1 \notin C$. The colors between two consecutive segments is called a *gap* of C. As we are working on a circular color palette (i.e. modulo k), the number of gaps is the same as the number of segments.

Let C be a proper subset of $\{0, 1, 2, \cdots, k-1\}$. A color j is called *attaching* to C if $j+1$ or $j-1$ belongs to C. A color j is called a *filling* color of C if both $j+1$ and $j-1$ belong to C. Denote by $A(C)$ and $F(C)$, respectively, the set of attaching colors and the set of filling colors of C.

Proposition 8 *Let C be a proper subset of* $\{0, 1, 2, \cdots, k-1\} \pmod{k}$.

(1) If $x \in F(C) - C$, then $\{x\}$ is a (singleton) gap of C.

(2) If $x \in C - A(C)$, then $\{x\}$ is a (singleton) segment of C.

For all the proofs of Theorems 1, 3 and 4, we define a sequential labeling for G, according to the ordering $v_1, v_2, \cdots, v_n$ (except for the proof of Theorem 3 in which a slight modification is necessary).

Suppose that ϕ is a *partial labeling* for V_{t-1} (i.e., ϕ is an assignment of colors to V_{t-1} which can be extended to a k-$L(2,1)$-labeling for G). For any $b \geq t$, a color j is *legal* for v_b, if for any $u \in V_{t-1}$, the following hold:

- If $u \sim v_b$, then $j \notin \{\phi(u), \phi(u) \pm 1\}$; and
- If $d_G(u, v_b) = 2$, then $j \neq \phi(u)$.

At each step, we extend ϕ from V_{t-1} to V_t by assigning a legal color to v_t. A color is *forbidden* for v_t if it is not legal for v_t. We denote by $\mathrm{Forb}(v_t)$ the set of forbidden colors for v_t.

For $u \in V_{t-1}$, set

$$C(u,t) = \{\phi(u), \phi(u)+1, \phi(u)-1\} \cup \{\phi(v_j) : j < t, v_j \sim u\}.$$

Lemma 9 *Let G be an outerplanar graph. Suppose ϕ is a partial labeling for V_{t-1}. Then the following hold:*

(1) $\mathrm{Forb}(v_t) \subseteq C(m(v_t), t) \cup C(f(v_t), t)$.

(2) $C(m(v_t), t) \subseteq \{\phi(m(v_t)), \phi(m(v_t)) \pm 1, \phi(m(m(v_t))), \phi(f(m(v_t))), \phi(x)\}$, *where x is a possible colored sibling of v_t.*

(3) If $f(m(v_t)) = f(v_t)$, then the vertex x in (2) does not exist, and hence $|C(m(v_t), t)| \leq 5$.

(4) $|\mathrm{Forb}(v_t) - C(f(v_t), t)| \leq 5$.

(5) If $f(m(v_t)) = f(v_t)$, then $|\mathrm{Forb}(v_t) - C(f(v_t), t)| \leq 4$.

(6) If G is triangulated, then

$$C(m(v_t), t) - C(f(v_t), t) \subseteq \{\phi(m(v_t)) \pm 1, \phi(x)\},$$

and $|\mathrm{Forb}(v_t) - C(f(v_t), t)| \leq 3$, *where x is a possible colored sibling of v_t.*

(7) *If* G *is triangulated and* $f(m(v_t)) = f(v_t)$, *then* $|\text{Forb}(v_t) - C(f(v_t), t)| \leq 2$.

Proof. By the ordering $\prec$, the only possible colored neighbors of v_t are $f(v_t)$ and $m(v_t)$. Moreover, if u is a colored vertex for which $d_G(u, v_t) = 2$, then u is adjacent to either $f(v_t)$ or $m(v_t)$. Therefore, (1) is true.

If $v_i \sim m(v_t)$ and $i < t$, then v_i is either a parent of $m(v_t)$ or a sibling of v_t. Hence, (2) is true.

If $f(m(v_t)) = f(v_t)$ and x is a sibling of v_t, then $f(x) = m(m(v_t))$ and hence $v_t \prec x$, i.e., x is not colored yet. So, (3) holds.

Note that, since $f(v_t)$ is also a parent of $m(v_t)$, we have $\phi(f(v_t)) \in C(f(v_t), t) \cap \{\phi(m(m(v_t))), \phi(f(m(v_t)))\}$. By (1) and (2), $|C(m(v_t), t) - C(f(v_t), t)| \leq 5$. This verifies (4).

If $f(m(v_t)) = f(v_t)$, then (5) follows by (1 − 4).

If G is triangulated, then $\{\phi(m(v_t)), \phi(m(m(v_t))), \phi(f(m(v_t)))\} \subseteq C(f(v_t), t)$. Hence, (6) follows by (1, 2). Moreover, if $f(m(v_t)) = f(v_t)$, then (7) follows by (1 − 4, 6). ∎

Proof of Theorem 3) We first consider the case that G is triangulated. By Lemma 9 (6), $|C(m(v_t), t) - C(f(v_t), t)| \leq 3$. As $|C(f(v_t), t)| \leq \Delta + 2$, by Lemma 9 (1), $|\text{Forb}(v_t)| \leq \Delta + 5$. If we had $\Delta + 6$ colors, then we would always had a legal color for v_t. However, we are given only $k = \Delta + 5$ colors. So our aim is to reduce the number of forbidden colors of v_t by 1. To accomplish this, we define a sequential coloring scheme such that the following property R1 is satisfied, at each step.

> **R1.** If $t \geq 3$, then $\phi(v_t)$ is an attaching color of either $C(f(v_t), t)$ or $C(m(v_t), t)$.

Note that, the coloring scheme is based upon a slight modification of the ordering $\prec$: We follow the ordering $\prec$, except when we encounter a pair of siblings, say x and y, then x and y are colored in an order depending on the color of their common minor parent.

Initially: Let $\phi(v_1) = 0$, $\phi(v_2) = 2$, $\phi(v_3) = 4$, so R1 is true.

Inductively: Suppose that ϕ has colored V_{t-1} such that R1 is satisfied at each step, and we want to color the vertex v_t. Assume v_t does not have a sibling. By Lemma 9 (6), we have $|\text{Forb}(v_t)| \leq |C(f(v_t), t)| + 2 \leq \Delta + 4$. As we are given $\Delta + 5$ colors, there exists some $j \in A(\text{Forb}(v_t)) - \text{Forb}(v_t)$. Let $\phi(v_t) = j$. Then R1 is satisfied.

Assume v_t has a sibling x. Then we first need to determine the order that we color v_t and x. Let $m(x) = m(v_t) = v_j$ for some $v_j \in V_{t-1}$. Assume that $f(x) = f(v_j)$ and $f(v_t) = m(v_j)$ (the other case, $f(x) = m(v_j)$ and $f(v_t) = f(v_j)$, can be proved similarly). By inductive hypothesis, $\phi(v_j)$ is attaching to either $C(m(v_j), j)$ or $C(f(v_j), j)$.

If $\phi(v_j)$ is attaching to $C(m(v_j), j)$, then we color x before v_t, by a legal color attaching to $\text{Forb}(x)$. This can be done, because x has no colored sibling and hence $|\text{Forb}(x)| \leq \Delta + 4$ (by Lemma 9 (6), and $|C(f(v_t), t)| \leq \Delta + 2$). Next, we find a legal color for v_t. Because $m(v_j) = f(v_t)$, and $\phi(v_j)$ is attaching to $C(m(v_j), j)$, we conclude that at least one of $\phi(v_j) + 1$ and $\phi(v_j) - 1$ is in $C(f(v_t), t)$. By Lemma 9 (6), one has $|C(m(v_t), t) - C(f(v_t), t)| \leq 2$, and so $|\text{Forb}(v_t)| \leq \Delta + 4$. Hence, there is a legal color for v_t that is attaching to $\text{Forb}(v_t)$, and R1 is satisfied.

If $\phi(v_j)$ is attaching to $C(f(v_j), j)$, then we label v_t before x. The discussion is the same as in the previous paragraph. This completes the proof for the existence of a labeling with at most $\Delta + 5$ colors for a triangulated outerplanar graph.

The second part of Theorem 3 for outerplanar graphs can be proved similarly, using (4), instead of (6), of Lemma 9. We omit the routineness. ∎

Suppose ϕ is a partial labeling for V_t. For any $u \in V_t$, set

$$C[u, t] = \begin{cases} C(u, t), & \text{if } v_t \not\sim u, \\ C(u, t) \cup \{\phi(v_t)\}, & \text{if } v_t \sim u. \end{cases}$$

Observe, $s[u, t] = |C[u, t]| - 3$.

Proof of Theorem 4) Let G be an outerplanar graph with $\Delta \geq 11$. Let $k = \Delta + 4$. Similar to the proof of Theorem 3, we give a sequential labeling scheme on the ordering $V(G) = \{v_1, v_2, \cdots, v_n\}$, using colors from the set $\{0, 1, 2, \cdots, k-1\}$. With fewer colors, we need to be more restrictive in bounding the size of $\mathrm{Forb}(v_t)$.

Suppose ϕ is a partial k-$L_c(2,1)$-labeling for V_t, where $t \geq 3$. Let $w = f(v_t)$, and let β be the number of segments in $C[w,t]$. Observe that

$$\beta \leq s[w,t] + 1. \tag{3.1}$$

We call ϕ a *valid partial labeling* for V_t if all the following hold:

R1. $\beta \leq 5$.

R2. If $s[w,t] \geq 5$, then $\phi(v_t) \in C[w,t]$.

R3. If $s[w,t] \geq 9$, then $\phi(v_t) \in A(C[w,t]) \cap C[w,t]$

We shall prove that for any $1 \leq t \leq n$, there is a valid partial labeling for V_t.

Initially: Let $\phi(v_1) = 0$, $\phi(v_2) = 2$, and $\phi(v_3) = 4$. Then R1 is true, while R2, R3 are vacuous.

Inductively: Assume ϕ is a valid partial labeling for V_{t-1}, $t \geq 4$. We extend ϕ to V_t, by assigning a color to v_t, so that ϕ is a valid partial labeling for V_t.

Assume $s[w,t] \leq 4$. Then $|C(w,t)| \leq 7$ and $|\mathrm{Forb}(v_t)| \leq 12$ (by Lemma 9 (1, 4)). Let $\phi(v_t) = j$ for some $j \notin \mathrm{Forb}(v_t)$ (j exists because $k = \Delta + 4 \geq 15$). By (3.1), R1 holds. R2 and R3 are vacuous.

Assume $s[w,t] \geq 5$. We consider two cases.

Case 1. $v_t \not\sim w$. Then $C(w,t) = C[w,t]$ no matter what legal color will be assigned to v_t. Moreover, we have

$$\mathrm{Forb}(v_t) \subseteq \{\phi(m(v_t)), \phi(m(v_t)) \pm 1, \phi(m(m(v_t))), \phi(f(m(v_t))), \phi(x)\},$$

where x is a possible already colored sibling of v_t. So, $|\mathrm{Forb}(v_t)| \leq 6$. Let q be the largest index such that $q < t$ and $v_q \sim w$. Then $s[w,q] = s[w,t] \geq 5$

and $C(w,t) = C[w,t] = C[w,q]$. By Lemma 7 (1), $f(v_q) = w$. So R1 holds by inductive hypothesis for V_q. Now, we need label v_t with a legal color so that R2 and R3 hold.

If $5 \le s[w,t] < 9$, then $|C(w,t)| \ge 8$, so there exists some $j \in C(w,t) - \mathrm{Forb}(v_t)$. Let $\phi(v_t) = j$. Then R2 holds, while R3 is vacuous.

If $s[w,t] \ge 9$, then $|C(w,t)| = |C[w,q]| \ge 12$. By inductive hypothesis for R1 on V_q, $C[w,q] = C(w,t)$ has at most 5 segments. By Prop. 8, $|C(w,t) - A(C(w,t))| \le 4$. Since $|C(w,t)| = |C[w,t]| = s[w,t] + 3 \ge 12$, we have $|A(C(w,t)) \cap C(w,t)| \ge 8$. As $|\mathrm{Forb}(v_t)| \le 6$, there exists some $j \in A(C(w,t)) \cap C(w,t)$ which is legal for v_t. Let $\phi(v_t) = j$. Then R2 and R3 hold, since $C(w,t) = C[w,t]$.

Case 2. $v_t \sim w.$ Then no matter what legal color is assigned to v_t, we have $|C(w,t)| = |C[w,t]| - 1 = s[w,t] + 2$, and $\phi(v_t) \in C[w,t]$. Thus, R2 holds always.

Assume $s[w,t] = 5$ or 6. Then $|C(w,t)| = 7$ or 8. By Lemma 9 (4), $|\mathrm{Forb}(v_t)| \le 13$. Because $k \ge 15$, there exists some color $j \notin \mathrm{Forb}(v_t)$. By inductive hypothesis, $C(w,t)$ has at most 5 segments. If $C(w,t)$ contains less than 5 segments, then let $\phi(v_t) = j$. So, R1 holds, while R3 is vacuous. Suppose $C(w,t)$ contains exactly 5 segments (so 5 gaps). Since $|C(w,t)| \le 8$ and $k \ge 15$, we conclude that there exists a gap with more than two elements, so $|A(C(w,t)) - C(w,t)| \ge 6$. By Lemma 9 (4), there exists some $j \in A(C(w,t)) - \mathrm{Forb}(v_t)$. Let $\phi(v_t) = j$. Then R1 holds, while R3 is vacuous.

Assume $s[w,t] \ge 7$. Let $v_{t'} = m(v_t)$. By Lemma 7 (2, 4), $f(v_{t'}) = w$ and $s[w,t'] \ge 5$. By Lemma 9 (5), $|\mathrm{Forb}(v_t) - C(w,t)| \le 4$. Moreover, by inductive hypothesis and R2, $\phi(v_{t'}) \in C[w,t'] \subseteq C(w,t)$. Therefore, we conclude that

$$\mathrm{Forb}(v_t) - C(w,t) \subseteq \{\phi(v_{t'}) \pm 1, \phi(m(v_{t'}))\}.$$

Assume $s[w,t] = 7, 8$. Then $|C(w,t)| = 9, 10$, and hence $|\mathrm{Forb}(v_t)| \le 13$. As $k \ge 15$, there exists some $j \notin \mathrm{Forb}(v_t)$. If $C(w,t)$ contains less than 5 segments, let $\phi(v_t) = j$. If $C(w,t)$ has exactly 5 segments (so 5 gaps), then by Prop. 8 and $k \geq 15$, we have $|A(C(w,t)) - C(w,t)| \geq 5$. Thus, there exists some $j \in A(C(w,t)) - \text{Forb}(v_t)$, as $|\text{Forb}(v_t) - C(w,t)| \leq 3$. Let $\phi(v_t) = j$. Then R1 holds, while R3 is vacuous.

Assume $9 \leq s[w,t] \leq \Delta - 1$. Then $s[w,t'] \geq 7$ and $|C(w,t)| \leq \Delta + 1$. By Lemma 7 (3), $w = f(m(v_{t'}))$. By inductive hypothesis and R2, $\{\phi(v_{t'}), \phi(m(v_{t'}))\} \subseteq C(w,t)$. So,

$$\text{Forb}(v_t) - C(w,t) \subseteq \{\phi(v_{t'}) \pm 1\}. \tag{3.2}$$

Because $|C(w,t)| \leq \Delta + 1$, $k = \Delta + 4$ and $\phi(v_{t'}) \in C(w,t)$, we conclude that $A(C(w,t)) - C(w,t) \not\subseteq \{\phi(v_{t'}) \pm 1\}$. Therefore, there exists a color $j \in A(C(w,t)) - \text{Forb}(v_t)$. Let $\phi(v_t) = j$. Then R1 and R3 hold.

Assume $s[w,t] = \Delta \geq 11$. Then $s[w,t'] \geq 9$. Similar to the above, (3.2) holds. Moreover, by R3, $\phi(v_{t'}) \in A(C(w,t')) \subset A(C(w,t))$, so one of $\phi(v_{t'}) \pm 1$ belongs to $C(w,t)$. Hence, $|\text{Forb}(v_t) - C(w,t)| \leq 1$ and $|\text{Forb}(v_t)| \leq \Delta + 3$ (because $|C(w,t)| \leq \Delta + 2$). Therefore, there exists some $j \in A(C(w,t)) - \text{Forb}(v_t)$. Let $\phi(v_t) = j$. Then R1 and R3 hold. ∎

4 Proof of Theorem 1 and Consequences

Similar to the previous section, we prove Theorem 1 by giving a sequential coloring scheme based upon the ordering $\prec$. Since we have fewer colors, the sequential coloring scheme is more restrictive. We use the same notations as in the previous section. Let $k = \Delta + 3$, and assume ϕ is a partial k-$L_c(2,1)$-labeling for V_t, where $t \geq 3$. Let $w = f(v_t)$, and let β be the number of segments in $C[w,t]$. If w has degree Δ, then let u be its Δ-th neighbor; otherwise u does not exist and we simply ignore the parts involving u. Throughout the proof we call ϕ a *valid partial labeling* for V_t if all the following hold:

R1. $\beta \leq 6$; and if $w \not\sim m(u)$ or $s[w,t] \leq 9$, then $\beta \leq 5$.

R2. If $s[w,t] \geq 5$, then $\phi(v_t) \in C[w,t]$.

R3. If $s[w,t] \geq 11$, then $\phi(v_t) \in C[w,t] \cap A(C[w,t])$.

R4. Assume w has degree Δ (i.e. u exists). If $s[w,t] \geq 10$ and $v_t \prec m(u)$, then there exists some $j^* \in F(C[w,t])$, which is legal for $m(u)$. Moreover, if $w \not\sim m(u)$, then

$$j^* \in C[w,t] \cap F(C[w,t]) \quad \text{and} \quad j^* \neq \phi(w).$$

We prove that for any $3 \leq t \leq n$, there is a valid partial labeling for V_t.

Initially: Let $\phi(v_1) = 0$, $\phi(v_2) = 2$ and $\phi(v_3) = 4$. Then R1 is true, while R2 - R4 are vacuous.

Inductively: Assume $t \geq 4$, and ϕ is a valid partial labeling for V_{t-1}.

If $s[w,t] \leq 4$, then $|C(w,t)| \leq 7$, and $|\text{Forb}(v_t)| \leq 12$ (by Lemma 9 (4)). Let $\phi(v_t) = j$ for some $j \notin \text{Forb}(v_t)$ (j exists because $k = \Delta + 3 \geq 18$). Then, R1 follows by (3.1), while R2 - R4 are vacuous.

Assume $s[w,t] \geq 5$. We consider two cases.

Case 1. $v_t \not\sim w$. Then $C(w,t) = C[w,t]$ no matter what legal color is assigned to v_t. Let q be the largest index such that $q < t$ and $v_q \sim w$. Then $s[w,q] = s[w,t] \geq 5$ and $C[w,t] = C(w,t) = C[w,q]$. By Lemma 7 (1), $w = f(v_q)$, and hence by inductive hypothesis, $C[w, v_q]$ has at most 5 segments. So, R1 holds.

Because $v_t \not\sim w$, the following follows by Lemma 9 (1, 2):

$$\text{Forb}(v_t) \subseteq \{\phi(m(v_t)), \phi(m(v_t)) \pm 1, \phi(m(m(v_t))), \phi(f(m(v_t))), \phi(x)\},$$

where x is a possible already colored sibling of v_t. (Note, $\phi(f(v_t))$ might be a forbidden color for v_t, however, as $f(v_t)$ is a parent of $m(v_t)$, so $\phi(f(v_t))$ is included in the set on the right-side above.) Hence, $|\text{Forb}(v_t)| \leq 6$.

Assume $5 \leq s[w,t] \leq 9$. Then $|C(w,t)| = s[w,t] + 3 \geq 8$. Hence, $|C(w,t) - \text{Forb}(v_t)| \geq 2$. Let $\phi(v_t) = j$ for some $j \in C(w,t) - \text{Forb}(v_t)$. Then R2 holds, while R3 and R4 are vacuous.

Assume $s[w,t] = s[w,q] \geq 10$. If u exists and $v_t \prec m(u)$, then by inductive hypothesis (applied to $C[w,v_q]$), there exists some $j^* \in F(C[w,q]) = F(C(w,t))$ which is legal for $m(u)$. We need to find a legal color for v_t, so that j^* is kept legal for $m(u)$.

If $s[w,t] = 10$, then $|C(w,t)| = 13$. As $|\text{Forb}(v_t)| \leq 6$, there exists some $j \in C(w,t) - (\text{Forb}(v_t) \cup \{j^*, j^* \pm 1\})$ (note: if u does not exist, we regard $\{j^*, j^* \pm 1\} = \emptyset$). Let $\phi(v_t) = j$. Then, j^* is still legal for $m(u)$, and R2, R4 hold (the "Moreover" part of R4 follows by inductive hypothesis and $C[w,q] = C[w,t]$), while R3 is vacuous.

Assume $s[w,t] \geq 11$. Then $|C(w,t)| \geq 14$. By Lemma 7, $f(m(m(v_t))) = f(m(v_t)) = w$. By Lemma 9 (3), $|\text{Forb}(v_t)| \leq 5$. By inductive hypothesis, $C(w,t)$ has at most 6 segments, and by definition, at most 5 of them are singletons. By Prop. 8, we have $|C(w,t) \cap A(C(w,t))| \geq 9$, implying that

$$|\,(C(w,t) \cap A(C(w,t))) - \{j^*, j^* \pm 1\}| \geq 6.$$

As $|\text{Forb}(v_t)| \leq 5$, there exists some $j \in C(w,t) \cap A(C(w,t)) - \{j^*, j^* \pm 1\} - \text{Forb}(v_t)$. Let $\phi(v_t) = j$ if $v_t \neq m(u)$; and $\phi(v_t) = j^*$ if $v_t = m(u)$. Then, R2 - R4 hold, since $C[w,t] = C(w,t) = C[w,q]$.

Case 2. $v_t \sim w$. Then R2 holds, regardless what color is assigned to v_t. It suffices to find a legal color for v_t that satisfies R1, R3 and R4.

Assume $5 \leq s[w,t] \leq 9$. Then, R3 and R4 are vacuous. Since $s[w,t] = |C[w,t]| - 3 = |C(w,t)| - 2$, one has $|C(w,t)| \leq 11$. By Lemma 9 (4), $|\text{Forb}(v_t)| \leq 16$. By inductive hypothesis, $C(w,t)$ has at most 5 segments. If $C(w,t)$ has less than 5 segments, let $\phi(v_t) = j$ for some $j \notin \text{Forb}(v_t)$, so R1 holds. If $C(w,t)$ has exactly 5 segments (so 5 gaps). Since $|C(w,t)| \leq 11$, and $k \geq 18$, we conclude that there exists a gap with at least two elements, so, $|A(C(w,t)) - C(w,t)| \geq 6$. By Lemma 9 (4), there exists some $j \in A(C(w,t)) - \text{Forb}(v_t)$. Let $\phi(v_t) = j$. Then R1 holds.

Assume $s[w,t] \geq 10$. Let $v_{t'} = m(v_t)$ and $v_{t''} = m(v_{t'})$. Then $t', t'' < t$. By Lemma 7, $s[w,t'] \geq 8$, $s[w,t''] \geq 6$ and $w = f(v_{t'}) = f(v_{t''})$. By inductive hypothesis and R2, we have $\{\phi(v_{t'}), \phi(v_{t''})\} \subseteq C(w,t)$. Therefore, by Lemma

9 (1, 2, 3, 5), we have:

$$\text{Forb}(v_t) - C(w,t) \subseteq \{\phi(v_{t'}) \pm 1\}. \qquad (4.1)$$

Lemma 10 *Let ϕ be a valid partial coloring for V_{t-1}. If $v_t \sim w$, $s[w,t] \geq 10$, and $|C(w,t)| \leq 15$, then $A(C(w,t)) - \text{Forb}(v_t) \neq \emptyset$.*

Proof. Assume $|C(w,t)| \leq 15$. For any segment $\{j, j+1, \cdots, j'\}$ of $C(w,t)$, we have $\{j-1, j'+1\} \subseteq A(C(w,t)) - C(w,t)$ (note, $j-1 \neq j'+1$ since $k \geq 18$). If $C(w,t)$ has more than one segment, then $|A(C(w,t)) - C(w,t)| \geq 3$. By (4.1), $A(C(w,t)) - \text{Forb}(v_t) \neq \emptyset$.

Assume $C(w,t)$ has only one segment, say $C(w,t) = \{j, j+1, \cdots, j'\}$. Then $A(C(w,t)) - C(w,t) = \{j-1, j'+1\}$. As $\phi(v_{t'}) \in C(w,t)$ (see above), it follows that $\{j-1, j'+1\} \neq \{\phi(v_{t'})) \pm 1\}$. So, $A(C(w,t)) - \text{Forb}(v_t) \neq \emptyset$. ∎

If u exists, then $u = v_b$, $m(u) = v_{b'}$ and $m(v_{b'}) = v_{b''}$ for some $b'' < b' < b$, and $s[w,b] = \Delta \geq 15$. By Lemma 7, $s[w,b'] \geq 13$, $s[w,b''] \geq 11$, and $f(v_{b'}) = f(v_{b''}) = w$.

Assume $s[w,t] = 10$. Then R3 is vacuous, $s[w,t-1] \leq 9$ and $|C(w,t)| = 12$. We consider two sub-cases.

Sub-case A. $s[w,t] = 10$ and, u does not exist or $m(u) \not\sim w$. By Lemma 10, there exists some $j \in A(C(w,t)) - \text{Forb}(v_t)$.

Let $\phi(v_t) = j$. Then R1 holds by inductive hypothesis. If u does not exists, then R4 is vacuous, and we are done.

Assume u exists and $m(u) = v_{b'} \not\sim w$. It suffices to verify R4, that is, to find some $j^* \in F(C[w,t]) \cap C[w,t] - \{\phi(w)\}$ such that j^* is legal for $v_{b'}$. As $s[w, v_{b''}] \geq 11$, so $v_t \prec v_{b''}$ (i.e. $v_{b''}$ has not been colored yet). Because $v_{b'} \not\sim w$, for j^* to be legal for $v_{b'}$, it suffices that $j^* \notin \{\phi(w), \phi(m(v_{b''}))\}$. Note, any segment of $C[w,t]$ has at most two ends, and all the colors in the segment except the ends are in $F(C[w,t])$. Because $C[w,t]$ has at most 5 segments, and $|C[w,t]| = 13$, we have $|C[w,t] \cap F(C[w,t])| \geq 3$. Hence, there exists some $j^* \in C[w,t] \cap F(C[w,t]) - \{\phi(w), \phi(m(v_{b''}))\}$, such that j^* is legal for $m(u)$. So R4 is satisfied.

Sub-case B. $s[w,t]=10$, u exists, and $m(u)\sim w$. In contrast to Sub-case A, we first fix j^*, then label v_t such that R1, R2 and R4 are satisfied.

Suppose $C(w,t)$ contains a singleton gap $\{i\}$. That is, $i\in F(C(w,t))-C(w,t)$. Let $j^*=i$. We need to show there exists a legal color for v_t so that j^* is kept legal for $m(u)$. As $|C(w,t)|=12$ and $\{j^*\}$ is a gap of $C(w,t)$ (so $C(w,t)$ contains at least two segments), by (4.1) and an argument similar to the proof of Lemma 10, there exists some $j\in A(C(w,t))-\text{Forb}(v_t)-\{j^*\}$. Let $\phi(v_t)=j$. Then R1 holds by inductive hypothesis. Because $s[w,b']\geq 13$ and $s[w,t]=10$, by Lemma 7 (5), we have $v_t\not\sim v_{b'}$. Hence j^* is legal for $v_{b'}=m(u)$, and R4 holds.

Now, suppose that every gap in $C(w,t)$ contains at least two elements. Note, $C(w,t)$ contains at most 5 segments, as $s[w,t-1]\leq 9$. Combining (4.1) with the assumptions $|C(w,t)|=12$ and $k\geq 18$, we conclude that there exists a gap $\{j,j+1,\cdots,j+i\}$ of $C(w,t)$ such that $i\geq 1$, and $j+1$ is legal for v_t. Let $j^*=j$ and $\phi(v_t)=j+1$. Then, j^* satisfies R4. Moreover, $C[w,t]$ contains at most 6 segments. So R1 holds. This completes the proof for the case $s[w,t]=10$.

Assume $s[w,t]=11,12$, so $|C(w,t)|=13,14$. Assume u exists and $m(u)\sim w$. By inductive hypothesis of R4, there exists some $\{j^*\}\in F(C[w,t-1])$ which is legal for $m(u)$. Then j^* must be a singleton gap of $C[w,t-1]=C(w,t)$, since $m(u)\sim w$. This implies that $C(w,t)$ contains at least two gaps. We claim:

$$A(C(w,t))-C(w,t)-\{\phi(v_{t'})\pm 1,j^*\}\neq\emptyset. \tag{4.2}$$

If $C(w,t)$ has more than two gaps, then $|A(C(w,t))-C(w,t)|\geq 4$. Therefore, (4.2) holds. If $C(w,t)$ has exactly two gaps, then $|A(C(w,t))-C(w,t)-\{j^*\}|=2$, since $k\geq 18$. Note, $A(C(w,t))-C(w,t)-\{j^*\}\neq\{\phi(v_{t'})\pm 1\}$, as $\phi(v_{t'})\in C(w,t)$. So, (4.2) holds.

Let $\phi(v_t)=j$ for some $j\in A(C(w,t))-C(w,t)-\{\phi(v_{t'}))\pm 1,j^*\}$. This justifies R1 and R3 (since $\phi(v_t)\in C[w,t]$). Moreover, since $j\notin\{j^*,j^*\pm 1\}$ (as $\{j^*\pm 1\}\subseteq C(w,t)$), j^* is still a legal color for $m(u)$. So, R4 holds.

Now, assume that u does not exist, or u exists but $m(u)\not\sim w$. As

$|C(w,t)| \leq 14$, by Lemma 10, there exists some $j \in A(C(w,t)) - \text{Forb}(v_t)$. Let $\phi(v_t) = j$. Then, R1 and R3 hold. If u does not exist, then R4 is vacuous. If u exists but $m(u) \not\prec w$, then by inductive hypothesis of R4, $\{j^*, j^* \pm 1\} \subseteq C(w,t)$. So $j \notin \{j^*, j^* \pm 1\}$, and j^* is still a legal color for $m(u)$. Hence, R4 holds.

Assume $13 \leq s[w,t] \leq \Delta - 1$. Then $v_t \neq u$. As $s[w,t'] \geq 11$, by inductive hypothesis and R3, $\phi(v_{t'}) \in A(C[w,t']) \cap C[w,t'] \subseteq A(C(w,t)) \cap C(w,t)$. Combining this with (4.1), one has $|\text{Forb}(v_t) - C(w,t)| \leq 1$.

Suppose $v_t \prec m(u)$. If $s[w,t] \leq \Delta - 2$, then $|C(w,t)| \leq \Delta$. Because $k = \Delta + 3$, $|\text{Forb}(v_t) - C(w,t)| \leq 1$ and $j^* \in F(C(w,t))$, we conclude that there exists some $j \in A(C(w,t)) - \text{Forb}(v_t) - \{j^*\}$. Let $\phi(v_t) = j$. Then R1, R3 (since $\phi(v_t) \in C[w,t]$) and R4 hold.

If $s[w,t] = \Delta - 1$, then $m(u) \not\prec w$ (as $v_t \prec m(u)$), and $|C(w,t)| = \Delta + 1$. By Sub-case A and inductive hypothesis, $\{j^*, j^* \pm 1\} \subseteq C(w,t)$. Therefore, there exists some $j \in A(C(w,t)) - \text{Forb}(v_t)$, as $k = \Delta + 3$. Let $\phi(v_t) = j$. Then R1, R3 and R4 hold.

Suppose $v_t = m(u)$. Let $\phi(v_t) = j^*$. Then R1, R3 and R4 hold.

Suppose $m(u) \prec v_t$ and $v_t \neq u$. Then $s[w,t] = \Delta - 1$, $|C(w,t)| = \Delta + 1$, and $j^* \in C(w,t)$. Because $|\text{Forb}(v_t) - C(w,t)| \leq 1$, $|C(w,t)| = \Delta + 1$, and $k = \Delta + 3$, there exists some $j \in A(C(w,t)) - \text{Forb}(v_t)$. Let $\phi(v_t) = j$. Then R1, R3 and R4 hold.

Assume $s[w,t] = \Delta \geq 15$. Then, $v_t = u$. By inductive hypothesis, $\phi(v_{t'}) = j^* \in F(C(w,t)) \cap C(w,t)$. By (4.1), one has $\text{Forb}(v_t) = C(w,t)$. As $|C(w,t)| = \Delta + 2$ and $k = \Delta + 3$, we conclude that there is a legal color for v_t. This completes the proof of the validity of the coloring scheme. ∎

The following corollary follows from (1.1) and Theorems 1, 3 and 4.

Corollary 11

$$\lambda(G) \leq \begin{cases} \Delta + 2, & \textit{if } G \textit{ is outerplanar with } \Delta(G) \geq 15; \\ \Delta + 3, & \textit{if } G \textit{ is outerplanar with } \Delta(G) \geq 11; \\ \Delta + 6, & \textit{if } G \textit{ is outerplanar}; \\ \Delta + 4, & \textit{if } G \textit{ is triangulated outerplanar}. \end{cases}$$

For outerplanar graphs with small maximum degrees, the equality of Theorem 1 does not always hold.

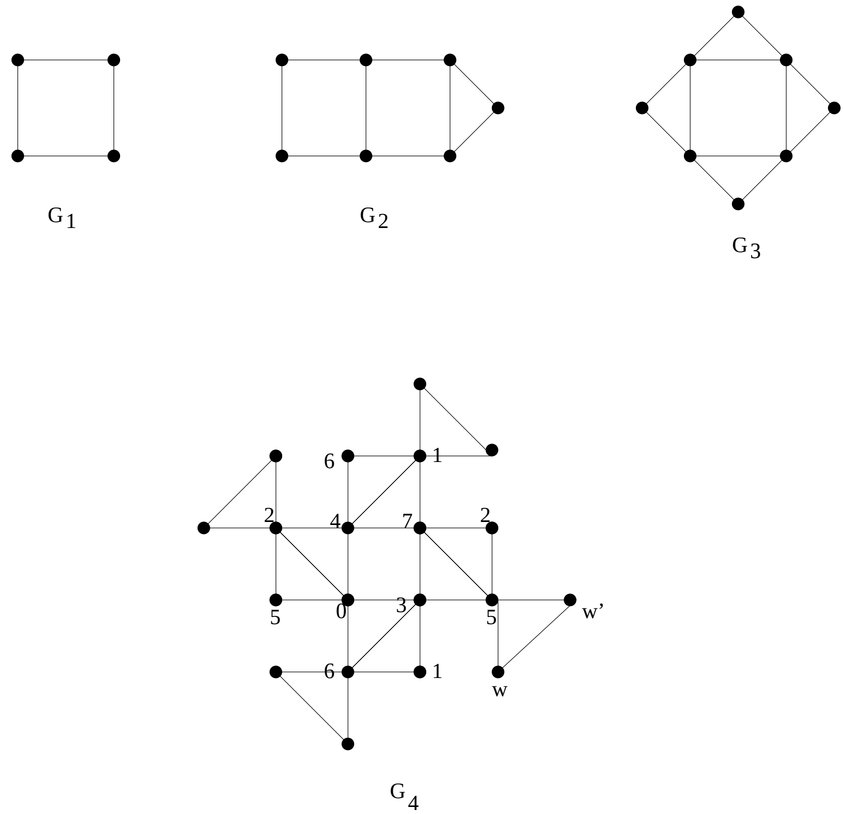

Figure 1: Graphs G_1, G_2, G_3, G_4

Theorem 12 *Let G_1, G_2, G_3, G_4 be the graphs as shown in Figure 1 above (ignore the labels of vertices of G_4 at the moment). Then $\Delta(G_i) = i + 1$, and $\lambda_c(G_i) = \Delta(G_i) + 4$.*

Proof. The proofs for G_1, G_2 and G_3, and for $\lambda_c(G_4) \leq \Delta(G_4) + 4$ are straightforward. It is more complicate and routine to verify that $\lambda_c(G_4) > \Delta(G_4) + 3 = 8$. One method to accomplish this is: 1) Prove that, by considering several cases, the labels (see Figure 1) for the "middle" induced subgraph H form a unique 8-$L_c(2,1)$-labeling for H; then 2) show that this unique labeling cannot be extended to the vertices w and w'. We omit the routineness. ∎

Theorem 12 indicates that a condition like $\Delta(G) \geq 15$ is necessary for Theorem 1. Indeed, the authors of this article suspect that the condition might be replaced by $\Delta(G) \geq d$ for some $6 \leq d < 15$. Finding the smallest such integer d for Theorem 1 would be an interesting problem for further research.

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Appendix: Proof of Theorem 12 (for the referees' reference)

To prove $\lambda_c(G_4) > \Delta + 3$, we first consider the subgraph H depicted in Figure 2. We show that $\lambda_c(H) = \Delta + 3$, and there is only one (up to symmetry) optimal circular distance two labeling for H.

Let f be an optimal labeling for H, $f : V(H) \to \{0, 1, 2, \cdots, 7\}$. For any edge $e = uv \in E(H)$, define the *weight* $w(uv)$ to be the circular difference between $f(u)$ and $f(v)$ modular 8, that is,

$$w(uv) = \min\{|f(u) - f(v)|,\ 8 - |f(u) - f(v)|\}.$$

Note that, for any $e \in E(H)$, we have $2 \le w(e) \le 4$. Let e_1, e_2, e_3, e_4 be the edges as shown in Figure 2 (i).

Claim 1 *There exists some $j \in \{1, 2, 3, 4\}$, such that $w(e_j) = 2$.*

Proof. Assume to the contrary that $w(e_j) = 3, 4$ for $j = 1, 2, 3, 4$. Due to the symmetry of the colors and the symmetric structure of H, without loss of generality, we assume that $f(v_1) = 0$, and $f(u_1) = 3$ or 4. If $f(u_1) = 3$, referring to Figure 2 (i), the colors assigned to x and v_2 must be distinct, and have circular difference at least 2 to both 0 and 3. The only possibility is that $\{f(x), f(v_2)\} = \{5, 6\}$. This implies that $\{f(v_4), f(u_4)\} = \{2, 4\}$, and hence $w(e_4) = 2$, contradicting the assumption. The argument for the case $f(u_1) = 4$ is similar. ∎

Thus, without loss of generality, we assume $f(v_1) = 0$ and $f(u_1) = 2$. As the neighbors of v_1 in H are in the set $\{v_2, u_1, x, u_4, v_4\}$, one has

$$\{f(v_2), f(x), f(u_4), f(v_4)\} = \{3, 4, 5, 6\}.$$

Note that the circular difference between $f(v_4)$ and $f(u_4)$ must be at least 2. Hence, there are the following eight possible cases:

	$f(x)$	$f(v_2)$	$f(v_4)$	$f(u_4)$
Case 1	4	6	3	5
Case 2	4	6	5	3
Case 3	6	4	3	5
Case 4	6	4	5	3
Case 5	4	5	3	6
Case 6	4	5	6	3
Case 7	5	4	6	3
Case 8	5	4	3	6

It is a routine to verify that the coloring in Cases 1 - 7 cannot be extended to the whole graph H. We show Cases 1 and 5, and leave the others to the reader, as the arguments are similar.

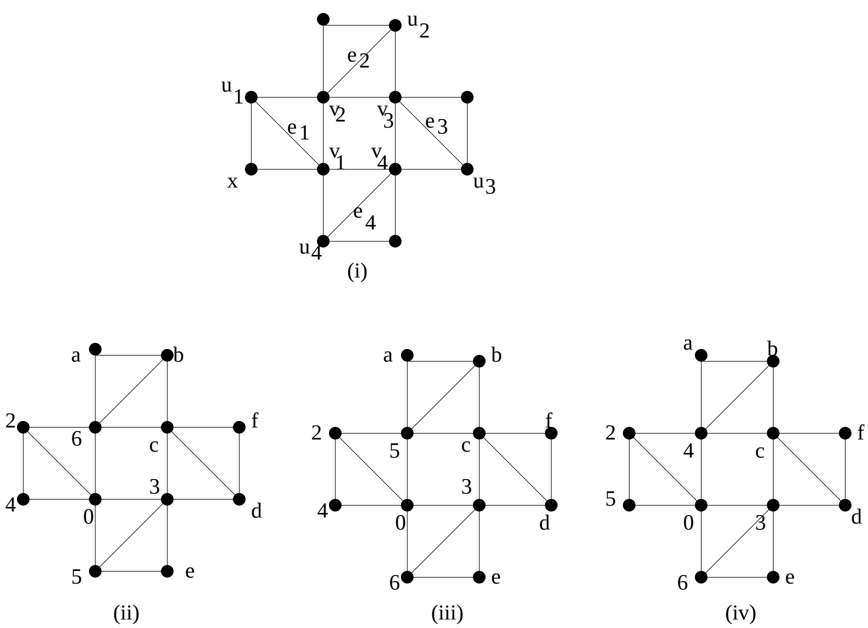

Figure 2: The subgraph H of G_4 and some possible labelings

For Case 1, the colors of the vertices are shown in Figure 2 (ii). By considering the neighbors of v_2, we conclude that $\{a, b, c\} = \{1, 3, 4\}$. This is impossible, because $f(v_4) = 3$, and the colors 3 and 4 cannot be assigned to adjacent vertices.

For Case 5, the colors of the vertices are shown in Figure 2 (iii). By considering the neighbors of v_4, we get $\{c, d, e\} = \{1, 5, 7\}$. This is impossible as apparently none of the c, d or e could be 5.

Next, we show that Case 8 can only be extended to a unique coloring for H. The colors of the vertices for Case 8 are shown in Figure 2 (iv). By considering the neighbors of v_2, we get $\{a, b, c\} = \{1, 6, 7\}$. By considering the neighbors of v_4, we conclude that $\{c, d, e\} = \{1, 5, 7\}$. But $c \neq 1$, for otherwise $\{a, b\} = \{6, 7\}$ which is impossible for adjacent vertices. Thus $c = 7$. This implies that $a = 6$, $b = 1$, $e = 1$, $d = 5$, and $f = 2$, which are shown in the center part of G_4 in Figure 1.

Assume there is a labeling for G_4 with span 8. Then this labeling must be an extension of the labels as depicted in Figure 1. However, this is impossible since it would imply that the colors for vertices w and w' must be $\{0, 1\}$, a contradiction. Therefore, we conclude that $\lambda_c(G_4) > \Delta + 3$.