**Complex Analysis Comprehensive Examination** **Fall 2021**

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Do five of the following seven problems. If you attempt more than 5, the best 5 will be used.

1. Evaluate

$$\int_{\gamma} \frac{1}{e^z - 1} dz$$

where $\gamma$ is the circle of radius 9 centered at 0.

2. a. Show that $|e^{-2z}| < 1$, if and only if, $\operatorname{Re} z > 0$.

b. Show that

$$\left| \int_{\gamma} \frac{e^{-2z}}{z} dz \right| < \frac{3}{\sqrt{5}}$$

where $\gamma$ is the line segment from $2 + i$ to $5 + i$.

3. For each of the following real valued functions of two variables $u(x, y)$, determine if there is a real valued function $v(x, y)$ such that $f(z) = f(x + iy) = u(x, y) + iv(x, y)$ is analytic. Either find $v(x, y)$, or explain why such function does not exist.

a. $u(x, y) = sinx - xy$ b. $u(x, y) = e^{-y} \sin x$

4. Find the Laurent series expansion for

$$f(z) = \frac{1}{z^2(1 - z)}$$

valid on in each of the regions $0 < |z| < 1$ , $1 < |z| < \infty$, and find the residue of $f(z)$ at $z_0 = 0$.

5. Suppose $n$ is a positive integer. Show there are exactly $n$ solutions counting multiplicity, to the equation $e^z = 4z^n - 1$ in the unit disk $|z| < 1$.

6. Consider the arcs $C_1$ defined by $z_1(t) = e^{it}$ where $0 \le t < \frac{3\pi}{2}$, and $C_2$ defined by $z_2(t) = t + i(t - 1)$ where $0 \le t \le 1$.

a. Draw the contour $C = C_1 + C_2$, and find its length.

b. Evaluate the integrals

$$\int_{C_1} \frac{dz}{z}, \quad \int_{C_2} \frac{dz}{z}, \quad \int_{C} \frac{dz}{z}.$$

7. Evaluate the following integrals by using residues:

a. $\int_0^\infty \frac{dx}{x^4+1}$

b. $\int_{-\infty}^\infty \frac{xdx}{(x^2+1)(x^2+2x+2)}$