California State University – Los Angeles
Department of Mathematics and Computer Science
Master's Degree Comprehensive Examination

Complex Analysis Fall 2000

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Do five of the following seven problems. Each problem is worth 20 points. Please write in a fairly soft pencil (number 2) (or in ink if you wish) so that your work will duplicate well. There should be a supply available.

Exams are being graded anonymously, so put your name only where directed and follow any instructions concerning identification code numbers.

Notation: $\mathbb{C}$ denotes the set of complex numbers.
$\mathbb{R}$ denotes the set of real numbers.
$\mathrm{Re}(z)$ denotes the real part of the complex number z.
$\mathrm{Im}(z)$ denotes the imaginary part of the complex number z.
$\overline{z}$ denotes the complex conjugate of the complex number z.
$|z|$ denotes the absolute value of the complex number z.
$\mathrm{Log}\, z$ denotes the principal branch of $\log z$.
$\mathrm{Arg}\, z$ denotes the principal branch of $\arg z$.
$D(z; r)$ is the open disk with center z and radius r.
A *domain* is an open connected subset of $\mathbb{C}$.

MISCELLANEOUS FACTS

$$2\sin a \sin b = \cos(a-b) - \cos(a+b) \qquad 2\cos a \cos b = \cos(a-b) + \cos(a+b)$$

$$2\sin a \cos b = \sin(a+b) + \sin(a-b) \qquad 2\cos a \sin b = \sin(a+b) - \sin(a-b)$$

$$\sin(a+b) = \sin a \cos b + \cos a \sin b \qquad \cos(a+b) = \cos a \cos b - \sin a \sin b$$

$$\tan(a+b) = \frac{\tan a + \tan b}{1 - \tan a \tan b}$$

$$\sin^2 a = \frac{1}{2} - \frac{1}{2}\cos(2a) \qquad \cos^2 a = \frac{1}{2} + \frac{1}{2}\cos(2a)$$

Fall 2000 #1. Suppose a and b are different real numbers. **Do either (A) or (B)**

(A) Show that the set $C = \left\{ z \in \mathbb{C} : \left| \frac{z-a}{z-b} \right| = 2 \right\}$ is a circle . Find the center and radius of that circle.

(OR)

(B) Show that the set $C = \left\{ z \in \mathbb{C} : \operatorname{Re}\left(\frac{z-a}{z-b} \right) = 0 \right\}$ is a circle (with one point deleted). Find the center and radius of that circle.

Fall 2000 #2. For z in $\mathbb{C}$, let $z = x + iy$ with x and y real. For each of the following real valued functions $u(x,y)$, determine whether there is a real valued function $v(x,y)$ such that the function $f(z) = u(x,y) + iv(x,y)$ is analytic and $f(0) = 1 + i$. If there is such a function v, find one and explain how you know that f is analytic. If there is not, explain how you know that there is not.

a. $u(x,y) = x^2 + e^x \cos y$

b. $u(x,y) = x + e^x \cos y$

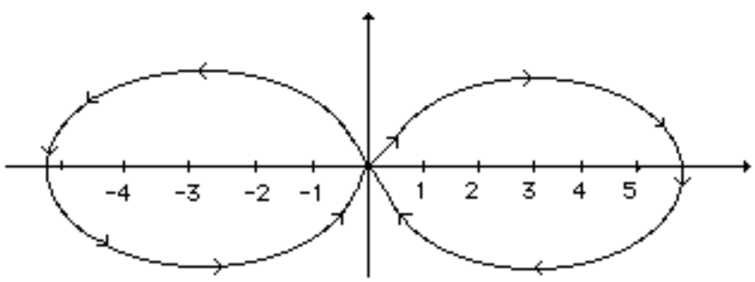
Fall 2000 #3. Evaluate the integral $\displaystyle\int_\gamma \frac{e^z\,dz}{z^2 - 2z - 15}$ Where γ is

(a) the circle of radius $\{z : |z| = 2\}$ traveled once counterclockwise.

(b) the circle of radius $\{z : |z| = 4\}$ traveled once counterclockwise.

(c) the circle of radius $\{z : |z| = 6\}$ traveled once counterclockwise.

(d) The "figure eight" shown in the sketch.



The curve γ for Problem 3d

Fall 2000 #4. Evaluate each of the following integrals. Explain contours and any estimates needed to justify your methods.

a. $\displaystyle\int_{-\pi}^{\pi} \frac{\cos\theta}{5 - 4\cos\theta}\,d\theta$ **b.** $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^2} e^{-itx}\,dx \quad t > 0.$

Fall 2000 #5. Let A be the annulus $A = \{z \in \mathbb{C} : 1/2 < |z| < 3/2\}$.

a. Find the Laurent series for the function $\dfrac{1}{(2z-1)(2z-3)}$ valid in A

b. Suppose $f(z)$ is analytic on A. For real θ, let $F(\theta) = f(e^{i\theta})$. Show that

$$F(\theta) = \sum_{n=-\infty}^{\infty} c_n e^{in\theta} \qquad \text{where} \qquad c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(\theta) e^{-in\theta}\, d\theta.$$

Fall 2000 #6. **a.** Find a conformal map $f(z)$ which maps the disk $D = \{z \in \mathbb{C} : |z-1| < 1\}$ one to one onto the upper half plane.

b. Find a confromal map $f(z)$ which maps the disk $D = \{z \in \mathbb{C} : |z-1| < 1\}$ one to one onto the set $W = \{z \in \mathbb{C} : 0 < \operatorname{Arg}(z) < \pi/4\}$.

Fall 2000 #7. Prove that the equation $e^z = 4z^4 + 1$ has 4 solutions in the disk $D = \{z \in \mathbb{C} : |z| \leq 1\}$ (counting possible multiplicity).

End of Exam