California State University – Los Angeles
Department of Mathematics
Master's Degree Comprehensive Examination

Complex Analysis Fall 2001

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Retyped with corrections: 2/24/02 Hoffman

Do five of the following seven problems. Each problem is worth 20 points. Please write in a fairly soft pencil (number 2) (or in ink if you wish) so that your work will duplicate well. There should be a supply available.

Exams are being graded anonymously, so put your name only where directed and follow any instructions concerning identification code numbers.

Notation: $\mathbb{C}$ denotes the set of complex numbers.
$\mathbb{R}$ denotes the set of real numbers.
$\mathrm{Re}(z)$ denotes the real part of the complex number z.
$\mathrm{Im}(z)$ denotes the imaginary part of the complex number z.
$\overline{z}$ denotes the complex conjugate of the complex number z.
$|z|$ denotes the absolute value of the complex number z.
$\mathrm{Log}\, z$ denotes the principal branch of $\log z$.
$\mathrm{Arg}\, z$ denotes the principal branch of $\arg z$.
$D(z;r)$ is the open disk with center z and radius r.
A *domain* is an open connected subset of $\mathbb{C}$.

MISCELLANEOUS FACTS

$$2\sin a \sin b = \cos(a-b) - \cos(a+b) \qquad 2\cos a \cos b = \cos(a-b) + \cos(a+b)$$
$$2\sin a \cos b = \sin(a+b) + \sin(a-b) \qquad 2\cos a \sin b = \sin(a+b) - \sin(a-b)$$
$$\sin(a+b) = \sin a \cos b + \cos a \sin b \qquad \cos(a+b) = \cos a \cos b - \sin a \sin b$$
$$\tan(a+b) = \frac{\tan a + \tan b}{1 - \tan a \tan b}$$
$$\sin^2 a = \frac{1}{2} - \frac{1}{2}\cos(2a) \qquad \cos^2 a = \frac{1}{2} + \frac{1}{2}\cos(2a)$$

Fall 2001 #1. **a** Let $f(z) = 1/z$. Show that the image of the set $\{z \in \mathbb{C} : \mathrm{Re}(z) = 1/2\}$ under the function f is a circle with one point missing. Find the center and radius of that circle. What is the missing point?

b. Express $\cos^5\theta$ as a linear combination of $\cos k\theta$ with $k = 0, 1, 2, 3, 4, 5$.

Fall 2001 #2. For which complex values of z do each of the following series converge? Give reasons for you answers.

$$\textbf{a.}\quad \sum_{n=1}^{\infty} \frac{z^n}{1-z^n} \qquad\qquad \textbf{b.}\quad \sum_{n=1}^{\infty} \frac{e^{nz}}{n^2}$$

Fall 2001 #3. Show that

$$\int_0^{2\pi} \cos^{2n}\theta\, d\theta = \frac{2\pi}{2^{2n}}\binom{2n}{n} = \frac{1\cdot 3\cdot 5\cdot \ldots \cdot (2n-1)}{2\cdot 4\cdot 6\cdot \ldots \cdot 2n} 2\pi.$$

Fall 2001 #4. Do two of the following three integrals. Show any contours and explain any estimates needed to justify your method.

a. Evaluate $\displaystyle\int_0^\infty \frac{1}{1+x^4}\,dx$

b. Show that $\displaystyle\int_0^\infty \frac{\ln x}{1+x^2}\,dx = \frac{\pi\ln 2}{4}$.

CORRECTION: This is wrong.

It should have been either $\displaystyle\int_0^\infty \frac{\ln x}{4+x^2}\,dx = \frac{\pi\ln 2}{4}$ or $\displaystyle\int_0^\infty \frac{\ln x}{1+x^2}\,dx = 0$. The first is probably better.

c. Evaluate $\displaystyle\int_0^\pi \frac{1}{5+3\cos\theta}\,d\theta$.

Fall 2001 #5. **a.** Determine the coefficients of the power series expansion $f(z) = \sum_{n=0}^\infty a_n z^n$ which satisfies the differential equation

$$zf''(z) + f'(z) + zf(z) = 0 \qquad \text{with } f(0) = 1.$$

(Note that a second initial condition, $f'(0) = 0$ comes automatically from the equation.)

b. Show that the resulting series converges to a function analytic on all of $\mathbb{C}$.

Fall 2001 #6. **a.** Show that if $f : \Omega \to \mathbb{C}$ is analytic on an open set Ω, then the real and imaginary parts of f must satisfy the Cauchy-Riemann equations.

b. Let $f : \mathbb{C} \to \mathbb{C}$ be analytic on all of $\mathbb{C}$ and suppose that $f(z+w) = f(z) + f(w)$ for all z and w in $\mathbb{C}$. Show that there is a constant a such that $f(z) = az$ for all z in $\mathbb{C}$.

(There are many ways to do this. Suggestion: What is $f(nz)$ for integer n ?)

Fall 2001 #7. Let $A = \{z \in \mathbb{C} : |z-1| < 1 \text{ and } |z-2| < 2\}$. Let C_1 and C_2 be the boundary circles $C_1 = \{z \in \mathbb{C} : |z-1| = 1\}$ and $C_2 = \{z \in \mathbb{C} : |z-2| = 2\}$.

a. (4 pts) Sketch the region A

b. (8 pts) Describe and sketch the image of the set A under the function $f(x) = 1/z$ giving reasons for your answer.

c. (8 pts) Find a function $u(x,y)$ which is harmonic except at $(0,0)$ with $u(x,y) = 2$ when $x+iy \in C_1 \setminus \{0\}$ and $u(x,y) = 1$ when $x+iy \in C_2 \setminus \{0\}$

SEMI-CORRECTION: This all right as written, but at least part **a** looks a little silly. The first inequality in the definition of A was typed backwards. It was intended to read $A = \{z \in \mathbb{C} : |z-1| > 1 \text{ and } |z-2| < 2\}$.

End of Exam