California State University – Los Angeles
Department of Mathematics
Master's Degree Comprehensive Examination
Complex Analysis Fall 2002

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Do five of the following seven problems. Each problem is worth 20 points. Please write in a fairly soft pencil (number 2) (or in ink if you wish) so that your work will duplicate well. There should be a supply available.

Exams are being graded anonymously, so put your name only where directed and follow any instructions concerning identification code numbers.

Notation: $\mathbb{C}$ denotes the set of complex numbers.
$\mathbb{R}$ denotes the set of real numbers.
$\mathrm{Re}(z)$ denotes the real part of the complex number z.
$\mathrm{Im}(z)$ denotes the imaginary part of the complex number z.
$\overline{z}$ denotes the complex conjugate of the complex number z.
$|z|$ denotes the absolute value of the complex number z.
$\mathrm{Log}\, z$ denotes the principal branch of $\log z$.
$\mathrm{Arg}\, z$ denotes the principal branch of $\arg z$.
$D(z; r)$ is the open disk with center z and radius r.
A *domain* is an open connected subset of $\mathbb{C}$.

Fall 2002 # 1. For z in $\mathbb{C}$, let $x = \mathrm{Re}(z)$ and $y = \mathrm{Im}(z)$.

For each of the following functions $u(x, y)$, determine whether there is a real valued function $v(x, y)$ such that $f(z) = f(x + iy) = u(x, y) + iv(x, y)$ is analytic with f(0) = 3i. If there is such a function v, find it. If there is not, explain how you know there is not.

a. $u(x, y) = (1 - x)^2 y$

b. $u(x, y) = (1 - x)y$

Fall 2002 # 2. Suppose the power series $\sum_{k=0}^{\infty} a_k z^k$ has radius of convergence 1. Call the value of the sum $f(z)$ and let $g(z) = f(z)/(1 - z)$.

a. Explain how you know $g(z)$ has a series representation of the form $g(z) = \sum_{n=0}^{\infty} b_n z^n$ valid for $|z| < 1$.

b. For each n find b_n in terms of a_0, a_1, a_2, a_3, $\ldots$.

Fall 2002 # 3. Suppose z and w are complex numbers with $|z| \leq 1$ and $|w| < 1$.
Show that $|z - w| \leq |1 - z\overline{w}| = |1 - \overline{z}w|$.
(Suggestion: Do it first with $|z| = 1$.)

Fall 2002 # 4. **a.** Suppose p and q are polynomials with degree$(q) \geq$ degree$(p) + 2$, and let $f(z) = p(z)/q(z)$.

Show that the sum of the residues of f at all of its singularities in $\mathbb{C}$ is equal to 0.

b. Let γ be the circle of radius 2 centered at 0 and travelled once counterclockwise. Evaluate $\displaystyle\int_\gamma \frac{1}{(z-3)(z^5-1)}\, dz$

Fall 2002 # 5. Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{\cos^2 x}{1+x^2}\, dx$.
Indicate curves and estimates used to justify your method.
(Suggestion: Write $\cos x$ in terms of exponentials.)

Fall 2002 # 6. **a. (5 points)** State a version of the Schwarz lemma.
b. (15 points) Let U be the open disk $D = \{z \in \mathbb{C} : |z| < 1\}$. Suppose $f : U \to U$ is analytic on U with $f(1/2) = 0$. Show that

$$|f(z)| \leq \left| \frac{z - (1/2)}{1 - (1/2)z} \right| = \left| \frac{2z-1}{2-z} \right|$$

for all z in U.

Fall 2002 # 7. Let $f(z) = z^4 + 3z + 1$
a. How many zeros, counting multiplicity, does f have in the annulus $A = \{z \in \mathbb{C} : 1 \leq |z| \leq 2\}$?
b. Can any of the zeros found in part **(a)** have multiplicity larger than 1?

End of Exam
