California State University – Los Angeles
Mathematics
Masters Degree Comprehensive Examination

Complex Analysis Fall 2009

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Do five of the following eight problems.
If you attempt more than 5, the best 5 will be used.
Please

(1) Please write in a fairly soft pencil (number 2) (or in ink if you wish) so that your work will duplicate well. There should be a supply available.
(2) Write on one side of the paper only
(3) Begin each problem on a new page
(4) Assemble the problems you hand in in numerical order

Exams are being graded anonymously, so put your name only where directed and follow any instructions concerning identification code numbers.

Notation: $\mathbb{C}$ denotes the set of complex numbers.
$\mathbb{R}$ denotes the set of real numbers.
$\mathrm{Re}(z)$ denotes the real part of the complex number z.
$\mathrm{Im}(z)$ denotes the imaginary part of the complex number z.
$\overline{z}$ denotes the complex conjugate of the complex number z.
$|z|$ denotes the absolute value of the complex number z.
$\mathrm{Log}\, z$ denotes the principal branch of $\log z$.
$\mathrm{Arg}\, z$ denotes the principal branch of $\arg z$.
$D(z; r)$ is the open disk with center z and radius r.
A *domain* is an open connected subset of $\mathbb{C}$.

MISCELLANEOUS FACTS

$$2\sin a \sin b = \cos(a-b) - \cos(a+b) \qquad 2\cos a \cos b = \cos(a-b) + \cos(a+b)$$

$$2\sin a \cos b = \sin(a+b) + \sin(a-b) \qquad 2\cos a \sin b = \sin(a+b) - \sin(a-b)$$

$$\sin(a+b) = \sin a \cos b + \cos a \sin b \qquad \cos(a+b) = \cos a \cos b - \sin a \sin b$$

$$\tan(a+b) = \frac{\tan a + \tan b}{1 - \tan a \tan b}$$

$$\sin^2 a = \frac{1}{2} - \frac{1}{2}\cos(2a) \qquad \cos^2 a = \frac{1}{2} + \frac{1}{2}\cos(2a)$$

Fall 2009 # 1. Describe and sketch each of the following sets of points in $\mathbb{C}$.

a. $A = \{z \in \mathbb{C} : |\, z - 1 \,| = 2\,|\, z \,|\}$.
b. $\mathrm{Re}(z^2) \geq 0$
c. $\mathrm{Im}(z^2) \geq 2$

Fall 2009 # 2. Suppose f is an entire function and that $\mathrm{Re}(f(z)) = \mathrm{Im}(f(z))$ for every z in $\mathbb{C}$. Show that f must be constant.

Fall 2009 # 3. Suppose $w(x, y)$ is a real valued function of two real variables x and y such that $w(1, 2) = 3$ and the function

$$f(z) = f(x + iy) = 2x - 4xy + 3y + iw(x, y)$$

is an analytic function of z. Find $w(2, 3)$.

Fall 2009 # 4. Consider the function $f(z) = \dfrac{z^5 + \sin(2z)}{z^6}$.

a. Find all singularities of f in $\mathbb{C}$ and classify each as removable, a pole (specify the order), essential, or other. (Give reasons for your answer.)
b. Evaluate $\int_\gamma f(z)\,dz$ where γ is the circle of radius 1 centered at 0 travelled once in the counterclockwise direction.

Fall 2009 # 5. Evaluate two of the following three integrals. Show contours and discuss estimates needed to justify your method.

a. $\displaystyle\int_{-\infty}^{\infty} \frac{1}{x^6 + 1}\,dx$ **b.** $\displaystyle\int_{-\infty}^{\infty} \frac{e^{x/2}}{1 + e^x}\,dx$ **c.** $\displaystyle\int_{0}^{\infty} \frac{\cos 2x}{1 + x^4}\,dx$

(In part **b** you might want to use the rectangle proceeding from $-R$ to R along the real axis, then up to $R + 2\pi i$, from there to $-R + 2\pi i$, and finally back to $-R$.)

Fall 2009 # 6. Let $f(z) = z^5 + 3z + 1$ and $A = \{z \in \mathbb{C} : 1 < |\, z \,| < 2\}$.

a. Counting each zero with its multiplicity (order), how many zeros does f have in the annulus A?
b. Can any of the zeros of f in A have multiplicity (order) larger than 1? (Justify your answer.)

Fall 2009 # 7. What is wrong with the following argument other than the absurd conclusion. You should be able to discuss at least one error in each numbered line.

For complex z consider the function $f(z) = \dfrac{\sin z}{z^2+1}$

(1) If $|z| \geq 2$, then $\left|z^2+1\right| \geq |z|^2 - 1 \geq 3$ so that
$$|f(z)| = \frac{|\sin z|}{|z^2+1|} \leq \frac{|\sin z|}{3} \leq \frac{1}{3}.$$

(2) Since $|f(z)| \leq 1/3$ on the circle of radius 2 centered at the origin, we also have $|f(z)| \leq 1/3$ for $|z| < 2$ by the maximum modulus principle.

(3) We now have $|f(z)| \leq 1/3$ for all z in $\mathbb{C}$, so f must be constant on $\mathbb{C}$ by Liouville's theorem.

Since f is constant and $f(\pi) = (\sin \pi)/(\pi^2+1) = 0$, we have $f(z) = 0$ for all z in $\mathbb{C}$. In particular, $\sin x = 0$ for all real x.

Fall 2009 # 8. Consider the regions

$H = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$
$Q = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0 \text{ and } \mathrm{Re}(z) > 0\}$
$D = \{z \in \mathbb{C} : |z| < 1\}$

For each of the following, decide whether an analytic function $f : A \to B$ taking A one-to-one onto B is possible. If it is, find such a function. If it is not, explain how you know it is not possible.

a. $f_a : \mathbb{C} \to D$ **b.** $f_b : D \to H$ **c.** $f_c : H \to Q$ **d.** $f_d : \mathbb{C} \to H$

(Hint: two are possible and two are not.)

End of Exam