California State University – Los Angeles
Mathematics
Masters Degree Comprehensive Examination

Complex Analysis   Fall 2010

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Do five of the following eight problems.
If you attempt more than 5, the best 5 will be used.
Please

(1) Please write in a fairly soft pencil (number 2) (or in ink if you wish) so that your work will duplicate well. There should be a supply available.
(2) Write on one side of the paper only
(3) Begin each problem on a new page
(4) Assemble the problems you hand in in numerical order

Exams are being graded anonymously, so put your name only where directed and follow any instructions concerning identification code numbers.

Notation: $\mathbb{C}$ denotes the set of complex numbers.
$\mathbb{R}$ denotes the set of real numbers.
$\text{Re}(z)$ denotes the real part of the complex number z.
$\text{Im}(z)$ denotes the imaginary part of the complex number z.
$\overline{z}$ denotes the complex conjugate of the complex number z.
$|z|$ denotes the absolute value of the complex number z.
$\text{Log}\, z$ denotes the principal branch of $\log z$.
$\text{Arg}\, z$ denotes the principal branch of $\arg z$.
$D(z; r)$ is the open disk with center z and radius r.
A *domain* is an open connected subset of $\mathbb{C}$.

MISCELLANEOUS FACTS

$2\sin a \sin b = \cos(a-b) - \cos(a+b)$ $2\cos a \cos b = \cos(a-b) + \cos(a+b)$

$2\sin a \cos b = \sin(a+b) + \sin(a-b)$ $2\cos a \sin b = \sin(a+b) - \sin(a-b)$

$\sin(a+b) = \sin a \cos b + \cos a \sin b$ $\cos(a+b) = \cos a \cos b - \sin a \sin b$

$$\tan(a+b) = \frac{\tan a + \tan b}{1 - \tan a \tan b}$$

$\sin^2 a = \frac{1}{2} - \frac{1}{2}\cos(2a)$ $\cos^2 a = \frac{1}{2} + \frac{1}{2}\cos(2a)$

Fall 2010 # 1. Describe and sketch each of the following sets

a. $A = \{z \in \mathbb{C} : \big(\mathrm{Re}(z)\big)^2 + 1 = \mathrm{Re}\left((z+1)^2\right)\}$
b. $B = \{z \in \mathbb{C} : \sin z$ is a real number $\}$

Fall 2010 # 2. For each of the following, classify the singularity of f at the specified point as a pole (what order?), removable, essential, or other. Also find the residue of f at that point.

$$\textbf{a. } f(z) = \frac{e^{1/z}}{z+1} \quad \text{at } z = -1 \qquad \textbf{b. } f(z) = \frac{e^{1/z}}{z} \quad \text{at } z = 0$$

$$\textbf{c. } f(z) = \frac{\sin(z^2)}{z^2} \quad \text{at } z = 0$$

Fall 2010 # 3. a. (4 points) State the Cauchy-Riemann equations for a complex-valued function f on $\mathbb{C}$

b. (8 points) Show that if $f : U \to \mathbb{C}$ is analytic (that is, the derivative exists as the limit of a difference quotient) on an open set U in $\mathbb{C}$, then the Cauchy-Riemann equations for f hold in U.

c. (8 points) Suppose $f : \mathbb{C} \to \mathbb{C}$ is analytic on $\mathbb{C}$ and that $f(z)$ is a real number for every z in $\mathbb{C}$. Show that f must be constant on $\mathbb{C}$.

Fall 2010 # 4. a. Suppose u and v are real valued functions on $\mathbb{C}$. Show that if v is a harmonic conjugate for u, then $-u$ is a harmonic conjugate for v.

b. Verify that $w(z) = \mathrm{Im}(z + e^z)$ is harmonic and find its harmonic conjugate.

Fall 2010 # 5. Find a conformal map of the half disk $A = \{z \in \mathbb{C} : |z| < 1,\ \mathrm{Im}\ z > 0\}$, onto the upper half-plane $H = \{z \in \mathbb{C} :\ \mathrm{Im}\ z > 0\}$. Suggestion: Combine a linear fractional transformation with the mapping $w \mapsto w^2$, and make sure you cover the whole upper half-plane.

Fall 2010 # 6. Evaluate each of the following integrals. Sketch curves and discuss estimates needed to justify your methods

$$\textbf{a. } \int_{-\infty}^{\infty} \frac{\cos(5x)}{x^2+9}\,dx \qquad \textbf{b. } \int_{-\infty}^{\infty} \frac{1+x^2}{4+x^4}\,dx$$

Fall 2010 # 7. Let $f(z) = \dfrac{7}{z^2 + z - 12}$.

a. Find the Laurent series for $f(z)$ valid for $3 < |\,z\,| < 4$.
b. What is the residue of f at 0?

Fall 2010 # 8. Evaluate the integral $\displaystyle\int_C \frac{e^{z^2}}{(z-2i)^2}\,dz$ for each of the following curves.

a. C_a is the cicle of radius 1 centered at the origin and travelled once counterclockwise.
b. C_b is the cicle of radius 3 centered at the origin and travelled once counterclockwise.

End of Exam