California State University – Los Angeles
Mathematics
Masters Degree Comprehensive Examination

Complex Analysis   Fall 2014

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Do five of the following seven problems.
If you attempt more than 5, the best 5 will be used.
Please

(1) Please write in a fairly soft pencil (number 2) (or in ink if you wish) so that your work will duplicate well. There should be a supply available.
(2) Write on one side of the paper only
(3) Begin each problem on a new page
(4) Assemble the problems you hand in in numerical order

Exams are being graded anonymously, so put your name only where directed and follow any instructions concerning identification code numbers.

Notation: $\mathbb{C}$ denotes the set of complex numbers.
$\mathbb{R}$ denotes the set of real numbers.
$\mathrm{Re}(z)$ denotes the real part of the complex number z.
$\mathrm{Im}(z)$ denotes the imaginary part of the complex number z.
$\overline{z}$ denotes the complex conjugate of the complex number z.
$|z|$ denotes the absolute value of the complex number z.
$\mathrm{Log}\, z$ denotes the principal branch of $\log z$.
$\mathrm{Arg}\, z$ denotes the principal branch of $\arg z$.
$D(z; r)$ is the open disk with center z and radius r.
A *domain* is an open connected subset of $\mathbb{C}$.

MISCELLANEOUS FACTS

$$2\sin a \sin b = \cos(a-b) - \cos(a+b) \qquad 2\cos a \cos b = \cos(a-b) + \cos(a+b)$$

$$2\sin a \cos b = \sin(a+b) + \sin(a-b) \qquad 2\cos a \sin b = \sin(a+b) - \sin(a-b)$$

$$\sin(a+b) = \sin a \cos b + \cos a \sin b \qquad \cos(a+b) = \cos a \cos b - \sin a \sin b$$

$$\tan(a+b) = \frac{\tan a + \tan b}{1 - \tan a \tan b}$$

$$\sin^2 a = \frac{1}{2} - \frac{1}{2}\cos(2a) \qquad \cos^2 a = \frac{1}{2} + \frac{1}{2}\cos(2a)$$

Fall 2014 # 1. Suppose $f : \mathbb{C} \to \mathbb{C}$ is not constant and is analytic on all of $\mathbb{C}$. Show:

a. There is at least one z in $\mathbb{C}$ with $|\, f(z)\,| > 1$.
b. There is at least one z in $\mathbb{C}$ with $|\, f(z)\,| < 1$.
c. There is at least one z in $\mathbb{C}$ with $|\, f(z)\,| = 1$.

(You might want to use both f and $1/f$.)

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Fall 2014 # 2. Evaluate the integral $\int_\gamma \frac{1}{(z-2)(z+4)}\, dz$ around each of the following curves. Give reasons for your answers.

a. The circle of radius 1 centered at 0 travelled once counterclockwise.
b. The circle of radius 3 centered at 0 travelled once counterclockwise.
c. The circle of radius 5 centered at 0 travelled once counterclockwise.
d. The path following straight line segments from $5-i$ to $5+2i$ to $-5+2i$ to -5 - 2i to $3-2i$ to $3+i$ to $1+i$ to $1-i$ and returning to $5-i$.

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Fall 2014 # 3. For z in $\mathbb{C}$, let $z = x + iy$ with x and y in $\mathbb{R}$. For each of the following real valued functions $u(x, y)$, determine whether there is a real valued function $v(x, y)$ such that $f(z) = u(x, y) + iv(x, y)$ is analytic with $f(0) = 1 + i$. If there is such a function, find one. If there is not, explain how you know there is not.

a. $u(x, y) = x^3 - 3xy^2 - y + 1$
b. $u(x, y) = x^2 - 3xy^2 - y + 1$

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Fall 2014 # 4. Evaluate each of the following integrals. Show any contours and discuss any estimates needed to justify your method.

a. $\displaystyle\int_0^\infty \frac{x^2}{(x^2+1)^2}\, dx$ **b.** $\displaystyle\int_0^\pi \frac{1}{5+3\cos\theta}\, d\theta$

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Fall 2014 # 5. For real t with $-1 < t < 1$ and z in $\mathbb{C}$, let $f(t, z)$ be defined by $f(t, z) = (1 - 2tz + z^2)^{-1}$.

a. Explain why $f(t, z)$ has an expansion of the form

$$f(t, z) = \frac{1}{1 - 2tz + z^2} = \sum_{n=0}^{\infty} U_n(t) z^n$$

b. Compute $U_0(t)$, $U_1(t)$, and $U_2(t)$ in terms of t.

c. Recalling that t is a real number smaller than 1 in absolute value, find the radius of convergence of this power series. (Hint: where are the singularities of $f(t, z)$ as a function of z?)

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Fall 2014 # 6. Let $f(z) = z^5 \cos(1/z)$.

a. Find the Laurent series for $f(z)$ valid for z near but not equal to 0. For what z is this expansion valid?

b. Evaluate $\int_\gamma f(z)\,dz$ with γ the circle of radius 1 centered at the origin and traveled once counterclockwise.

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Fall 2014 # 7. Let C_1 be the circle of radius 1 centered at 1.
Let C_2 be the circle of radius 2 centered at 2.
Let A be the region between the two circles.

a. Show that the function $f(z) = 1/z$ maps the set A onto the vertical strip $S = \{w \in \mathbb{C} : 1/4 < \mathrm{Re}(w) < 1/2\}$.

b. Find a function $\phi(x, y)$ such that ϕ is harmonic on A and continuous on $A \cup C_1 \cup C_2$ except at 0 with $\phi(x, y) = 2$ for (x, y) on $C_1 \setminus \{0\}$ and $\phi(x, y) = 1$ for (x, y) on $C_2 \setminus \{0\}$. Say something to justify your answer.

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End of Exam