California State University – Los Angeles
Mathematics
Masters Degree Comprehensive Examination

Complex Analysis   Fall 2016

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Do five of the following seven problems.
If you attempt more than 5, the best 5 will be used.
Please

(1) Please write in a fairly soft pencil (number 2) (or in ink if you wish) so that your work will duplicate well. There should be a supply available.
(2) Write on one side of the paper only
(3) Begin each problem on a new page
(4) Assemble the problems you hand in in numerical order

Exams are being graded anonymously, so put your name only where directed and follow any instructions concerning identification code numbers.

Notation: $\mathbb{C}$ denotes the set of complex numbers.
$\mathbb{R}$ denotes the set of real numbers.
$\mathrm{Re}(z)$ denotes the real part of the complex number z.
$\mathrm{Im}(z)$ denotes the imaginary part of the complex number z.
$\overline{z}$ denotes the complex conjugate of the complex number z.
$|z|$ denotes the absolute value of the complex number z.
$\mathrm{Log}\, z$ denotes the principal branch of $\log z$.
$\mathrm{Arg}\, z$ denotes the principal branch of $\arg z$.
$D(z; r)$ is the open disk with center z and radius r.
A *domain* is an open connected subset of $\mathbb{C}$.

MISCELLANEOUS FACTS

$$2\sin a\sin b = \cos(a-b) - \cos(a+b) \qquad 2\cos a\cos b = \cos(a-b) + \cos(a+b)$$

$$2\sin a\cos b = \sin(a+b) + \sin(a-b) \qquad 2\cos a\sin b = \sin(a+b) - \sin(a-b)$$

$$\sin(a+b) = \sin a\cos b + \cos a\sin b \qquad \cos(a+b) = \cos a\cos b - \sin a\sin b$$

$$\tan(a+b) = \frac{\tan a + \tan b}{1 - \tan a\tan b}$$

$$\sin^2 a = \frac{1}{2} - \frac{1}{2}\cos(2a) \qquad \cos^2 a = \frac{1}{2} + \frac{1}{2}\cos(2a)$$

Fall 2016 # 1. Find all the singularities of $f(z) = \dfrac{z^2+3z}{(z+2)^3(z^2-9)}$ in $\mathbb{C}$, and classify each as removable, a pole (specify the order), or essential.

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Fall 2016 # 2. Suppose $f : D \to \mathbb{C}$ is analytic on the disk $D = \{z \in \mathbb{C} : |\, z\,| < 1\}$. Let $x = \mathrm{Re}(z)$, $y = \mathrm{Im}(z)$, $u(x,y) = \mathrm{Re}(f(x+iy))$, and $v(x,y) = \mathrm{Im}(f(x+iy))$. Show that if $u(x,y) + v(x,y) = 17$ everywhere in D, then $f(z)$ must be constant on D.

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Fall 2016 # 3. Let γ be the closed path consisting of straight line segments from $3+3i$ to $-3-3i$, from there to $-3+3i$, from there to $3-3i$, and finally back to $3+3i$. Evaluate $\int_\gamma f(z)\,dz$ for each of the following functions giving reasons for your answers.

$$\textbf{a.}\quad f(z) = \frac{1}{z^2-4} \qquad \textbf{b.}\quad f(z) = \frac{\sin(\pi z)}{(z-1)^2}$$

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Fall 2016 # 4. For each of the following real valued functions $u(x,y)$, decide whether there can be another real valued function $v(x,y)$ such that $f(x+iy) = u(x,y) + iv(x,y)$ is analytic (at least on some open subset of $\mathbb{C}$). If "yes", find such a function. If "no" explain how you know that there can be no such function.

$$\textbf{a.}\quad u(x,y) = y^3 - 2x^2y \qquad \textbf{b.}\quad u(x,y) = y^3 - 3x^2y$$

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Fall 2016 # 5. Let A be the closed unit disk $A = \{z \in \mathbb{C} : |\, z\,| \le 1\}$.
Suppose f is an entire function whose Taylor series centered at the origin is $\sum_{k=0}^{\infty} a_k z^k$, and that f maps A into A.
Show that $|\, a_k\,| \le 1$ for each k.

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Fall 2016 # 6. Evaluate **TWO** of the following three integrals. Show any paths and discuss any estimates needed to justify your method. The path in part **c** is the circle of radius 1 centered at 0 travelled once counterclockwise.

$$\textbf{a.}\quad \int_{-\infty}^{\infty} \frac{x^2}{x^4+16}\,dx \qquad \textbf{b.}\quad \int_0^{\pi} \frac{1}{3+2\cos t}\,dt \qquad \textbf{c.}\quad \int_\gamma z^3 e^{2/z}\,dz$$

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Fall 2016 # 7. For real t let $f_t(z) = \dfrac{ze^{tz}}{e^z - 1}$. One way to define the Bernoulli polynomials $B_k(t)$ is by

$$f_t(z) = \sum_{k=0}^{\infty} \frac{B_k(t)}{k!}\, z^k$$

a. Explain in terms of singularities of f_t how you know that $f_t(z)$ has an expansion of this form.
b. Compute $B_0(t)$, $B_1(t)$, and $B_2(t)$

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End of Exam