California State University – Los Angeles
Department of Mathematics and Computer Science
Master's Degree Comprehensive Examination

Complex Analysis   Spring 2001

Hoffman, Katz*, Kolesnik

Do five of the following seven problems. Each problem is worth 20 points. Please write in a fairly soft pencil (number 2) (or in ink if you wish) so that your work will duplicate well. There should be a supply available.

Exams are being graded anonymously, so put your name only where directed and follow any instructions concerning identification code numbers.

Notation: $\mathbb{C}$ denotes the set of complex numbers.
$\mathbb{R}$ denotes the set of real numbers.
$\mathrm{Re}(z)$ denotes the real part of the complex number z.
$\mathrm{Im}(z)$ denotes the imaginary part of the complex number z.
$\overline{z}$ denotes the complex conjugate of the complex number z.
$|z|$ denotes the absolute value of the complex number z.
$\mathrm{Log}\, z$ denotes the principal branch of $\log z$.
$\mathrm{Arg}\, z$ denotes the principal branch of $\arg z$.
$D(z;r)$ is the open disk with center z and radius r.
A *domain* is an open connected subset of $\mathbb{C}$.

Spring 2001 # 1. Describe or sketch each of the following sets in $\mathbb{C}$.

a. $A = \{z \in \mathbb{C} : |z-2| > |z-3|\}$

b. $B = \{z \in \mathbb{C} : 1/z = \overline{z}\}$

c. $C = \{z \in \mathbb{C} : \left|z^2\right| = \mathrm{Im}(z)\}$

d. $D = \{z \in \mathbb{C} : \left|z^2 - 1\right| < 1\}$ Suggestion: In part **(d)**, $z^2 = w$ lies in some disk. What disk? First sketch that disk, then sketch the set of all z which square into it.

Spring 2001 # 2. Suppose Ω is an open connected subset of $\mathbb{C}$. For z in Ω, let $z = x+iy$ with x and y real. For $f : \Omega \to \mathbb{C}$, let $u(x,y) = \mathrm{Re}(f(x+iy))$ and $v(x,y) = \mathrm{Im}(f(x+iy))$.

a. Show that if f is analytic on Ω, then u and v must satisfy the Cauchy-Riemann equations. (Derive those equations.)

b. Use the Cauchy-Riemann equations to show that if f is analytic on Ω and the image $f(\Omega)$ is contained in the diagonal line $v = u$, then $f(z)$ is constant on Ω.

Spring 2001 # 3. For real t with $-1 < t < 1$ and z in $\mathbb{C}$, let $f(t,z)$ be defined by $f(t,z) = (1 - 2tz + z^2)^{-1}$.

a. Explain why $f(t,z)$ has an expansion of the form

$$f(t,z) = \frac{1}{1 - 2tz + z^2} = \sum_{n=0}^{\infty} U_n(t) z^n$$

b. Compute $U_0(t)$, $U_1(t)$, and $U_2(t)$ in terms of t.

c. Recalling that t is a real number smaller than 1 in absolute value, show that the radius of convergence of this power series in z is 1. (Hint: where are the singularities of $f(t,z)$ as a function of z?)

Spring 2001 # 4. Evaluate $\displaystyle\int_\gamma \frac{z\,dz}{z^2 - 4z + 3}$ for each of the curves γ indicated.

a γ the circle of radius 1 centered at 1 travelled counterclockwise.

b γ the circle of radius 2 centered at 2 travelled counterclockwise.

c γ the polygonal path produced by following stright line segments from $-2i$ to $2i$ to $4 - 2i$ to $4 + 2i$, and finally back to $-2i$.

Spring 2001 # 5. Evaluate each of the following integrals

$$\textbf{a. } \int_0^\infty \frac{\cos 2x}{x^2 + 1}\,dx \qquad\qquad \textbf{b. } \int_0^\infty \frac{x^2}{x^4 + 1}\,dx$$

Show contours and indicate estimates needed to justify your method.

Spring 2001 # 6. **a.** Find the number of zeros (counting multiplicity) for the function $f(z) = z^6 + 3z + 1$ in the annulus $A = \{z \in \mathbb{C} : 1 \le |z| \le 2\}$.

b. Can any of these zeros have multiplicity larger than one? (Hint: What is true of f at a point which is a multiple zero?)

Spring 2001 # 7. Suppose $f : \mathbb{C} \to \mathbb{C}$ is a polynomial and that $f'(z)$ is never equal to 0. Show that f must be one-to-one on $\mathbb{C}$. Is this true for an arbitrary entire function ? (Prove or give a counterexample.)

End of Exam