California State University – Los Angeles
Department of Mathematics
Master's Degree Comprehensive Examination

Complex Analysis Spring 2004

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Do five of the following seven problems.

Each problem is worth 20 points. Please write in a fairly soft pencil (number 2) (or in ink if you wish) so that your work will duplicate well. There should be a supply available.

Exams are being graded anonymously, so put your name only where directed and follow any instructions concerning identification code numbers.

Notation: $\mathbb{C}$ denotes the set of complex numbers.
$\mathbb{R}$ denotes the set of real numbers.
$\mathrm{Re}(z)$ denotes the real part of the complex number z.
$\mathrm{Im}(z)$ denotes the imaginary part of the complex number z.
$\overline{z}$ denotes the complex conjugate of the complex number z.
$|z|$ denotes the absolute value of the complex number z.
$\mathrm{Log}\, z$ denotes the principal branch of $\log z$.
$\mathrm{Arg}\, z$ denotes the principal branch of $\arg z$.
$D(z; r)$ is the open disk with center z and radius r.
A *domain* is an open connected subset of $\mathbb{C}$.

MISCELLANEOUS FACTS

$$2\sin a \sin b = \cos(a-b) - \cos(a+b) \qquad 2\cos a \cos b = \cos(a-b) + \cos(a+b)$$

$$2\sin a \cos b = \sin(a+b) + \sin(a-b) \qquad 2\cos a \sin b = \sin(a+b) - \sin(a-b)$$

$$\sin(a+b) = \sin a \cos b + \cos a \sin b \qquad \cos(a+b) = \cos a \cos b - \sin a \sin b$$

$$\tan(a+b) = \frac{\tan a + \tan b}{1 - \tan a \tan b}$$

$$\sin^2 a = \frac{1}{2} - \frac{1}{2}\cos(2a) \qquad \cos^2 a = \frac{1}{2} + \frac{1}{2}\cos(2a)$$

Spring 2004 # 1. Let $p(z) = 1 + z^2 + z^4 + z^6 + z^8 + z^{10}$ and $f(z) = 1/p(z)$

a. Identify and sketch the set of zeros of the polynomial p.

b. Find the radius of convergence of the Taylor series for f centered at 1.

Suggestion: Consider $(1 - z^2)p(z)$.

Spring 2004 # 2. For positive integer n, let $(1 + z)^n = c_0 + c_1 z + c_2 z^2 + \cdots + c_n z^n$. Use standard complex analysis techniques to establish the binomial formula:

$$c_k = \binom{n}{k} = \frac{n!}{k!\,(n-k)!} \qquad \text{for } 0 \le k \le n.$$

Spring 2004 # 3. Let D be the open unit disk in $\mathbb{C}$. Prove or disprove each of the following statements.

a. There is a function $f : \mathbb{C} \to D$ which is analytic on $\mathbb{C}$ and which maps $\mathbb{C}$ onto D.

b. There is a function $g : D \to \mathbb{C}$ which is analytic on D and which maps D one-to-one onto $\mathbb{C}$.

c. There is a function $h : D \to \mathbb{C}$ which is analytic on D and which maps D onto $\mathbb{C}$.

Spring 2004 # 4. Evaluate $\displaystyle\int_\gamma \frac{e^z}{(z-2)(z-4)}\,dz$ for each of the following paths γ.

a. the circle of radius 1 centered at 0 traveled once counterclockwise

b. the circle of radius 3 centered at 0 traveled once counterclockwise

c. the circle of radius 5 centered at 0 traveled once counterclockwise

d. the polygonal path following straight line segments from $3i$ to $6-3i$ to $6+3i$ to $-3i$ and back to $3i$.

Spring 2004 # 5. Evaluate each of the following integrals. Show any curves and explain estimates needed to justify your method.

$$\textbf{a.} \quad \int_0^\pi \frac{dt}{4 + \cos t} \qquad\qquad \textbf{b.} \quad \int_{-\infty}^\infty \frac{dx}{x^4 + 2x^2 + 1}$$

Spring 2004 # 6. Suppose n is an integer and $n \ge 3$.

Show that all of the solutions of the equation $nz^n = 1 + z + z^2 + \cdots + z^n$ lie in the disk $\{z \in \mathbb{C} : |z| < 3/2\}$.

Spring 2004 # 7. Let $D = \{z \in \mathbb{C} : |z| < 2\}$. Suppose $f : D \setminus \{1\} \to \mathbb{C}$ is analytic on D except for a pole of order 1 at 1 with residue b at that point. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ for $|z| < 1$.

a. Show that $\lim_{n\to\infty} a_n = -b$.

(Suggestion: Write out the series for $h(z) = (z-1)f(z)$ centered at 0 in terms of the coefficients a_n and consider its partial sums evaluated at 1.)

a. Illustrate the result of part **a** using $f(z) = \dfrac{1}{z-2} - \dfrac{1}{z-1}$

(That is: Compute the series expansion for $f(z)$ centered at 0. Compute the residue of f at 1, and show that the coefficients of that series converge to the negative of that residue.)

End of Exam