California State University – Los Angeles
Department of Mathematics
Master's Degree Comprehensive Examination
Complex Analysis   Spring 2006
Akis, Cooper*, Hoffman, Katz

Do five of the following eight problems.
If you attempt more than 5, the best 5 will be used.
Please

(1) Please write in a fairly soft pencil (number 2) (or in ink if you wish) so that your work will duplicate well. There should be a supply available.
(2) Write on one side of the paper only
(3) Begin each problem on a new page
(4) Assemble the problems you hand in in numerical order

Exams are being graded anonymously, so put your name only where directed and follow any instructions concerning identification code numbers.

Notation: $\mathbb{C}$ denotes the set of complex numbers.
$\mathbb{R}$ denotes the set of real numbers.
$\text{Re}(z)$ denotes the real part of the complex number z.
$\text{Im}(z)$ denotes the imaginary part of the complex number z.
$\overline{z}$ denotes the complex conjugate of the complex number z.
$|z|$ denotes the absolute value of the complex number z.
$\text{Log}\, z$ denotes the principal branch of $\log z$.
$\text{Arg}\, z$ denotes the principal branch of $\arg z$.
$D(z; r)$ is the open disk with center z and radius r.
A *domain* is an open connected subset of $\mathbb{C}$.

MISCELLANEOUS FACTS

$$2\sin a \sin b = \cos(a-b) - \cos(a+b) \qquad 2\cos a \cos b = \cos(a-b) + \cos(a+b)$$
$$2\sin a \cos b = \sin(a+b) + \sin(a-b) \qquad 2\cos a \sin b = \sin(a+b) - \sin(a-b)$$
$$\sin(a+b) = \sin a \cos b + \cos a \sin b \qquad \cos(a+b) = \cos a \cos b - \sin a \sin b$$
$$\tan(a+b) = \frac{\tan a + \tan b}{1 - \tan a \tan b}$$
$$\sin^2 a = \frac{1}{2} - \frac{1}{2}\cos(2a) \qquad \cos^2 a = \frac{1}{2} + \frac{1}{2}\cos(2a)$$

Spring 2006 # 1. **a.** For which values of θ with $0 \le \theta < 2\pi$ does the limit $\lim_{r\to\infty} e^{re^{i\theta}}$ exist

i. as a finite value in $\mathbb{C}$?
ii. as a value in the extended complex plane $\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$(the Riemann sphere)?

b. Suppose $f : \mathbb{C} \to \mathbb{C}$ is analytic with $\mathrm{Re}(f(z)) = \mathrm{Im}(f(z))$ for all z in $\mathbb{C}$. Show that f is constant on $\mathbb{C}$.

(Note: Parts **a** and **b** are not related.)

Spring 2006 # 2. For each of the following real valued functions $u(x, y)$ of real variable x and y, determine whether there is a real valued function $v(x, y)$ such that the function $f(z) = f(x + iy) = u(x, y) + iv(x, y)$ is analytic. If "yes", find such a function v. If "no", explain how you know there is no such function.

a. $u(x, y) = 3x^2 - 3y^2 - y + 5$
b. $u(x, y) = 3x^2 + 3y^2 - y + 5$

Spring 2006 # 3. For complex z, let $z = x + iy$ with x and y real, and consider a function $f(z) = f(x+iy) = u(x, y)+iv(x, y)$ on $\mathbb{C}$ with u and v real valued functions having continuous partial derivatives.

Differential operators $\partial/\partial z$ and $\partial/\partial \overline{z}$ are defined by

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \qquad \text{and} \qquad \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

($\partial/\partial z$ and $\partial/\partial \overline{z}$ are not meant to be partial derivatives in the usual sense. They are suggestive notations for the operators defined by the expressions on the right side of the equations which are the usual sort of partial derivatives.)

a. What are the Cauchy-Riemann equations for f?
b. Assuming knowledge about the Cauchy-Riemann equations, show that f is analytic if and only if $\dfrac{\partial}{\partial \overline{z}} f = 0$
b. Assuming knowledge about the Cauchy-Riemann equations, show that if f is analytic then $\dfrac{\partial}{\partial z} f = f'$.

Spring 2006 # 4. Let $f(z) = \dfrac{\cot z}{z^2}$.

a. Explain why $f(z)$ has a series expansion of the form $f(z) = b_3 z^{-3} + b_2 z^{-2} + b_1 z^{-1} + a_0 + a_1 z + a_2 z^2 + \dots$ valid in a punctured disk $B = \{z \in \mathbb{C} : 0 < |z| < r\}$.
b. What is the largest possible value for r?
c. Find b_3, b_2, and b_1.
d. Evaluate $\int_\gamma f(z)\, dz$ where γ is the unit circle centered at the origin.

Spring 2006 # 5. Evaluate two of the following three integrals. Show any contours and explain any estimates needed to justify your methods.

a. $\displaystyle\int_{-\infty}^{\infty} \frac{1}{x^2-2x+2}\,dx$ **b.** $\displaystyle\int_{-\infty}^{\infty} \frac{\cos 2x}{x^2-2x+2}\,dx$ **c.** $\displaystyle\int_{0}^{2\pi} \frac{1}{5+3\sin\theta}\,d\theta$

(If you attempt all three, the best two will be used.)

Spring 2006 # 6. Show that the equation $e^z = 4z^n - 1$ has exactly n solutions, counting multiplicity, inside the unit circle (that is, with $|z| < 1$).

Spring 2006 # 7. Evaluate $\displaystyle\int_\gamma \frac{\sin \pi z}{(2z-3)(2z-5)}\,dz$ for each of the following curves γ

a. The circle of radius 1 centered at the origin and travelled once counterclockwise.
b. The circle of radius 2 centered at the origin and travelled once counterclockwise.
c. The circle of radius 3 centered at the origin and travelled once counterclockwise.
d. The path following straight line segments from $2i$ to $4-2i$ to $4+2i$ to $-2i$ and back to $2i$.

Spring 2006 # 8. Let $f(z) = \csc(1/z)$, and let $B = \{z \in \mathbb{C} : 0 < |z| < 1\}$. Show that $f(B)$ is dense in $\mathbb{C}$.

(That is: for each w in $\mathbb{C}$ and each $\varepsilon > 0$, there is a z in B with $|f(z) - w| < \varepsilon$.)
(If z_k is a pole of f, $f(z_k) = \infty$.)
(Suggestion: Think about the proof of the Casorati-Weierstrass theorem.)

End of Exam