California State University – Los Angeles
Department of Mathematics
Master's Degree Comprehensive Examination

Complex Analysis Spring 2007

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Do five of the following eight problems.
If you attempt more than 5, the best 5 will be used.
Please

(1) Please write in a fairly soft pencil (number 2) (or in ink if you wish) so that your work will duplicate well. There should be a supply available.
(2) Write on one side of the paper only
(3) Begin each problem on a new page
(4) Assemble the problems you hand in in numerical order

Exams are being graded anonymously, so put your name only where directed and follow any instructions concerning identification code numbers.

Notation: $\mathbb{C}$ denotes the set of complex numbers.
$\mathbb{R}$ denotes the set of real numbers.
$\mathrm{Re}(z)$ denotes the real part of the complex number z.
$\mathrm{Im}(z)$ denotes the imaginary part of the complex number z.
$\overline{z}$ denotes the complex conjugate of the complex number z.
$|z|$ denotes the absolute value of the complex number z.
$\mathrm{Log}\, z$ denotes the principal branch of $\log z$.
$\mathrm{Arg}\, z$ denotes the principal branch of $\arg z$.
$D(z;r)$ is the open disk with center z and radius r.
A *domain* is an open connected subset of $\mathbb{C}$.

MISCELLANEOUS FACTS

$$2\sin a\sin b = \cos(a-b) - \cos(a+b) \qquad 2\cos a\cos b = \cos(a-b) + \cos(a+b)$$

$$2\sin a\cos b = \sin(a+b) + \sin(a-b) \qquad 2\cos a\sin b = \sin(a+b) - \sin(a-b)$$

$$\sin(a+b) = \sin a\cos b + \cos a\sin b \qquad \cos(a+b) = \cos a\cos b - \sin a\sin b$$

$$\tan(a+b) = \frac{\tan a + \tan b}{1 - \tan a\tan b}$$

$$\sin^2 a = \frac{1}{2} - \frac{1}{2}\cos(2a) \qquad \cos^2 a = \frac{1}{2} + \frac{1}{2}\cos(2a)$$

Spring 2007 # 1. **a.** Find all solutions to the equation $z^3 = \overline{z}$.
b. Let $z_1 = 1 + i$ and $z_2 = -1 - i$. Find all complex numbers z_3 such that the triangle with vertices at z_1, z_2, z_3 is equilateral.

Spring 2007 # 2. Show that if n is an integer greater than or equal to 3 and $\zeta_0, \zeta_1, \ldots, \zeta_{n-1}$ are the n-th roots of 1, then $\displaystyle\sum_{k=0}^{n-1} \zeta_k^2 = 0$.

Spring 2007 # 3. Define a sequence a_0, a_1, a_2, … by setting $a_0 = 1$, $a_1 = 2$, and $a_n = (a_{n-1} + a_{n-2})/2$ for $n \geq 2$.

- **a.** Find the radius of convergence of the series $\displaystyle\sum_{n=0}^{\infty} a_n z^n$.
  (Suggestion: In what range are the coefficients a_n?)
- **b.** Find an explicit formula for the function $f(z)$ defined by the series of part **(a)**.
  (Suggestion: find a way to use the fact that $2a_n - a_{n-1} - a_{n-2} = 0$ for $n \geq 2$.)

Spring 2007 # 4. Suppose $f : \mathbb{C} \to \mathbb{C}$ is analytic on $\mathbb{C}$ with $|\,f(z)\,| \leq \sqrt{|\,z\,|}$ for all z in $\mathbb{C}$. Let $g(z) = f(z)/z$.

- **a.** Discuss the singularity of g at 0. (Give reasons for your conclusions.)
- **b.** Show that $f(z) = 0$ for all z in $\mathbb{C}$.

Spring 2007 # 5. Evaluate each of the following integrals.

- **a.** $\displaystyle\int_{-\infty}^{\infty} \frac{\cos ax}{x^2 + b^2}\, dx$ where a and b are positive real constants. Show contours and explain estimates needed to justify your method.
- **b.** $\displaystyle\int_{\gamma} z^5 e^{1/z}\, dz$ where γ is the circle of radius 1 centered at the origin and travelled once in the counterclockwise direction.

Spring 2007 # 6. Evaluate the integral $\displaystyle\int_{\gamma} \frac{e^{z/2}}{(z+2)(z-4)}\, dz$ for each of the following curves γ. Give reasons for your answers.

- **a.** γ the circle of radius 1 centered at the origin and travelled once in the counterclockwise direction.
- **b.** γ the circle of radius 3 centered at the origin and travelled once in the counterclockwise direction.
- **c.** γ the circle of radius 5 centered at the origin and travelled once in the counterclockwise direction.
- **d.** γ the polygonal path made by following line segments from $-3+3i$ to $5-5i$ to $5+5i$ to $-3-3i$ and finally back to $-3+3i$

Spring 2007 # 7. Let D be the open unit disk $\{z \in \mathbb{C} : |\,z\,| < 1\}$ and Q be the open first quadrant, $\{z \in \mathbb{C} : \mathrm{Re}(z) > 0 \text{ and } \mathrm{Im}(z) > 0\}$. Find a function f analytic on Q mapping Q one-to-one onto D with $f(1+i) = 0$.

Spring 2007 # 8. Suppose f is an entire function and $f(0) = 1 + i$. Let $u(x, y) = \mathrm{Re}(f(x + iy)$ and $v(x, y) = \mathrm{Im}(f(x + iy)$.

a. (4 points) State the Cauchy-Riemann equations for u and v.

b. (8 points) Show that the function u is a harmonic function of x and y.

c. (8 points) Show that the curves defined in the xy-plane by $u(x, y) = 1$ and $v(x, y) = 1$ cross at right angles at the origin.

End of Exam