California State University – Los Angeles
Department of Mathematics
Master's Degree Comprehensive Examination

Complex Analysis   Spring 2008

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Do five of the following seven problems.
If you attempt more than 5, the best 5 will be used.

Please

1. Please write in a fairly soft pencil (number 2) (or in ink if you wish) so that your work will duplicate well. There should be a supply available.
2. Write on one side of the paper only
3. Begin each problem on a new page
4. Assemble the problems you hand in in numerical order

Exams are being graded anonymously, so put your name only where directed and follow any instructions concerning identification code numbers.

Notation: $\mathbb{C}$ denotes the set of complex numbers.
$\mathbb{R}$ denotes the set of real numbers.
$\text{Re}(z)$ denotes the real part of the complex number z.
$\text{Im}(z)$ denotes the imaginary part of the complex number z.
$\overline{z}$ denotes the complex conjugate of the complex number z.
$|z|$ denotes the absolute value of the complex number z.
$\text{Log}\, z$ denotes the principal branch of $\log z$.
$\text{Arg}\, z$ denotes the principal branch of $\arg z$.
$D(z; r)$ is the open disk with center z and radius r.
A *domain* is an open connected subset of $\mathbb{C}$.

MISCELLANEOUS FACTS

$$2\sin a \sin b = \cos(a-b) - \cos(a+b) \qquad 2\cos a \cos b = \cos(a-b) + \cos(a+b)$$

$$2\sin a \cos b = \sin(a+b) + \sin(a-b) \qquad 2\cos a \sin b = \sin(a+b) - \sin(a-b)$$

$$\sin(a+b) = \sin a \cos b + \cos a \sin b \qquad \cos(a+b) = \cos a \cos b - \sin a \sin b$$

$$\tan(a+b) = \frac{\tan a + \tan b}{1 - \tan a \tan b}$$

$$\sin^2 a = \frac{1}{2} - \frac{1}{2}\cos(2a) \qquad \cos^2 a = \frac{1}{2} + \frac{1}{2}\cos(2a)$$

Spring 2008 # 1. For each of the following, describe and sketch the set of all complex numbers z for which the indicated relation is true.

a. $|\,z\,|^2 = \mathrm{Re}(z^2)$
b. $|\,z\,|^2 = \mathrm{Im}(z^2)$
c. $|\,z\,|^2 = (\arg z)^2$ (Here $0 \le \arg z < 2\pi$.)

Spring 2008 # 2. Evaluate $\int_\gamma \left(\frac{e^{2z}}{z-2} + \frac{e^{3z}}{(z+5)^3} \right) dz$ for each of the following curves γ

a. The circle of radius 1 centered at 0 and travelled once counterclockwise.
b. The circle of radius 3 centered at 0 and travelled once counterclockwise.
c. The circle of radius 6 centered at 0 and travelled once counterclockwise.
d. The path formed by following straight line segments from $6+i$ to $-6-i$, from there to $-6+i$, then to $6-i$, and finally back to $6+i$.

Spring 2008 # 3. Let $D = \{z \in \mathbb{C} : |\,z\,| < 1\}$ be the open unit disk in the complex plane.

a. Find a function f which maps D one-to-one conformally onto the quarter plane $Q = \{z \in \mathbb{C} : \mathrm{Re}(z) > 0 \text{ and } \mathrm{Im}(z) > 0\}$.
b. Find a function g which maps D one-to-one conformally onto D with $g(1/2) = 1/3$.

Spring 2008 # 4. Show that all zeros of the polynomial $p(z) = z^6 - 5z^2 + 10$ lie in the annulus $A = \{z \in \mathbb{C} : 1 < |\,z\,| < 2\}$.

Spring 2008 # 5. Let $f(z) = \frac{1}{(z-1)(z-2)}$. Find the Laurent series for f valid in each of the following regions.

a. $\{z \in \mathbb{C} : |\,z\,| < 1\}$ **b.** $\{z \in \mathbb{C} : 1 < |\,z\,| < 2\}$ **c.** $\{z \in \mathbb{C} : |\,z\,| > 2\}$

Spring 2008 # 6. **a.** Use complex analysis to prove the fundamental theorem of algebra: If p is a nonconstant polynomial with coefficients in $\mathbb{C}$, then there is at least one point w in $\mathbb{C}$ with $p(w) = 0$.

b. Suppose $f : \mathbb{C} \to \mathbb{C}$ is analytic on all of $\mathbb{C}$, and $\left| f^{(5)}(z) \right| < 17$ for all z in $\mathbb{C}$. Show that f is a polynomial. What can you say about the degree of f?

Spring 2008 # 7. Evaluate each of the following integrals. Sketch any curves and discuss estimates needed to justify your method.

a. $\int_0^{2\pi} \frac{\sin^2 t}{5 + 4\cos t}\, dt$ **b.** $\int_0^{\infty} \frac{x^2+1}{x^4+1}\, dx$

End of Exam