California State University – Los Angeles
Mathematics
Masters Degree Comprehensive Examination

Complex Analysis Spring 2013

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Do five of the following seven problems.
If you attempt more than 5, the best 5 will be used.
Please

(1) Please write in a fairly soft pencil (number 2) (or in ink if you wish) so that your work will duplicate well. There should be a supply available.
(2) Write on one side of the paper only
(3) Begin each problem on a new page
(4) Assemble the problems you hand in in numerical order

Exams are being graded anonymously, so put your name only where directed and follow any instructions concerning identification code numbers.

Notation: $\mathbb{C}$ denotes the set of complex numbers.
$\mathbb{R}$ denotes the set of real numbers.
$\text{Re}(z)$ denotes the real part of the complex number z.
$\text{Im}(z)$ denotes the imaginary part of the complex number z.
$\overline{z}$ denotes the complex conjugate of the complex number z.
$|z|$ denotes the absolute value of the complex number z.
$\text{Log}\, z$ denotes the principal branch of $\log z$.
$\text{Arg}\, z$ denotes the principal branch of $\arg z$.
$D(z; r)$ is the open disk with center z and radius r.
A *domain* is an open connected subset of $\mathbb{C}$.

MISCELLANEOUS FACTS

$$2\sin a \sin b = \cos(a-b) - \cos(a+b) \qquad 2\cos a \cos b = \cos(a-b) + \cos(a+b)$$

$$2\sin a \cos b = \sin(a+b) + \sin(a-b) \qquad 2\cos a \sin b = \sin(a+b) - \sin(a-b)$$

$$\sin(a+b) = \sin a \cos b + \cos a \sin b \qquad \cos(a+b) = \cos a \cos b - \sin a \sin b$$

$$\tan(a+b) = \frac{\tan a + \tan b}{1 - \tan a \tan b}$$

$$\sin^2 a = \frac{1}{2} - \frac{1}{2}\cos(2a) \qquad \cos^2 a = \frac{1}{2} + \frac{1}{2}\cos(2a)$$

Spring 2013 # 1. Describe and sketch each of the following sets.

a. $A = \{z \in \mathbb{C} : z^2 + \overline{z}^2 = 2\}$ **b.** $B = \{z \in \mathbb{C} : \left| e^{(z^2)} \right| \leq e\}$

c. $C = \{z \in \mathbb{C} : \mathrm{Im}(2/z) < 1\}$

Spring 2013 # 2. Find the Laurent expansions for the function $\dfrac{4}{z^2 - 2x - 3}$ valid in each of the following regions

a. $0 < |\, z \,| < 1$ **b.** $1 < |\, z \,| < 3$ **c.** $|\, z \,| > 3$

Spring 2013 # 3. For each of the following, classify the singularity at the indicated point as removable, a pole (state the order of each pole), or essential and find the residue at that point.

a. $f(z) = \dfrac{e^z}{z^2 - 1}, \quad z_0 = 1$ **b.** $f(z) = \dfrac{1 - \cos z}{z^5}, \quad z_0 = 0$

c. $f(z) = \dfrac{\sin\left(z^2\right)}{z^2}, \quad z_0 = 0$ **d.** $f(z) = z^3 \sin(1/z), \quad z_0 = 0$

Spring 2013 # 4. Determine the number of zeros (counting multiplicity) of the polynomial $p(z) = z^7 - 4z^3 + z - 1$ in each of the following regions.

a. $A = \{z \in \mathbb{C} : |\, z \,| < 1\}$ **b.** $B = \{z \in \mathbb{C} : 1 < |\, z \,| < 2\}$

c. $C = \{z \in \mathbb{C} : |\, z \,| > 2\}$

Spring 2013 # 5. Evaluate each of the following integrals.

a. $\dfrac{1}{\pi} \displaystyle\int_{-\infty}^{\infty} \frac{g(x)}{1 + x^2}\, dx$ **b.** $\displaystyle\int_0^{2\pi} \sin^4 \theta \, d\theta$

In part (a), g is an entire function with $g(x) \in \mathbb{R}$ when $x \in \mathbb{R}$ and $|\, g(z) \,| \leq 1$ when $\mathrm{Im}\, z \geq 0$. The answer should be in terms of g. Be sure to show curves and discuss any inequalities needed to justify your method.

Spring 2013 # 6. a. Find the image of the interior of the circle $C = \{z \in \mathbb{C} : |\, z - 2 \,| = 2\}$ under the transfomation $z \mapsto w = f(z) = z/(2z - 8)$.

b. Find a function g which is analytic on the region $E = \{z \in \mathbb{C} : |\, z \,| > 1\}$ and maps E one-to-one onto $H = \{w \in \mathbb{C} : \mathrm{Re}\, w < 0\}$.

Spring 2013 # 7. Suppose we know the following about a function $f(z)$.

i. $f(z + 1) = f(z)$ and $f(z + i) = f(z)$ for all z in $\mathbb{C}$.
ii. f has only isolated singularities (if any) in $\mathbb{C}$
iii. f has no singularities on the boundary S of the square with corners at 0, 1, i, and $1 + i$.

Show that

a. If f has no singularities inside S, then f must be constant on $\mathbb{C}$.
b. If f has only one singularity inside S, then that singularity cannot be a pole of order 1. (Suggestion: Consider $\int_S f(z)\, dz$.)

End of Exam