California State University – Los Angeles
Mathematics
Masters Degree Comprehensive Examination

Complex Analysis   Spring 2016

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Do five of the following seven problems.
If you attempt more than 5, the best 5 will be used.
Please

(1) Please write in a fairly soft pencil (number 2) (or in ink if you wish) so that your work will duplicate well. There should be a supply available.
(2) Write on one side of the paper only
(3) Begin each problem on a new page
(4) Assemble the problems you hand in in numerical order

Exams are being graded anonymously, so put your name only where directed and follow any instructions concerning identification code numbers.

Notation: $\mathbb{C}$ denotes the set of complex numbers.
$\mathbb{R}$ denotes the set of real numbers.
$\mathrm{Re}(z)$ denotes the real part of the complex number z.
$\mathrm{Im}(z)$ denotes the imaginary part of the complex number z.
$\overline{z}$ denotes the complex conjugate of the complex number z.
$|z|$ denotes the absolute value of the complex number z.
$\mathrm{Log}\, z$ denotes the principal branch of $\log z$.
$\mathrm{Arg}\, z$ denotes the principal branch of $\arg z$.
$D(z; r)$ is the open disk with center z and radius r.
A *domain* is an open connected subset of $\mathbb{C}$.

MISCELLANEOUS FACTS

$$2\sin a \sin b = \cos(a-b) - \cos(a+b) \qquad 2\cos a \cos b = \cos(a-b) + \cos(a+b)$$

$$2\sin a \cos b = \sin(a+b) + \sin(a-b) \qquad 2\cos a \sin b = \sin(a+b) - \sin(a-b)$$

$$\sin(a+b) = \sin a \cos b + \cos a \sin b \qquad \cos(a+b) = \cos a \cos b - \sin a \sin b$$

$$\tan(a+b) = \frac{\tan a + \tan b}{1 - \tan a \tan b}$$

$$\sin^2 a = \frac{1}{2} - \frac{1}{2}\cos(2a) \qquad \cos^2 a = \frac{1}{2} + \frac{1}{2}\cos(2a)$$

Spring 2016 # 1. Describe and sketch each of the following subsets of $\mathbb{C}$.

a. $A = \{z \in \mathbb{C} : \operatorname{Re}(z^2) \geq 1\}$ **b.** $B = \{z \in \mathbb{C} : \operatorname{Im}(z^2) \geq 2\}$

c. $C = \{z \in \mathbb{C} : |\, z - 2\,| = |\, z - 4\,|\}$ **d.** $D = \{z \in \mathbb{C} : \overline{z} = 1/z\}$

Spring 2016 # 2. Evaluate each of the following integrals. In part **(a)**, γ is the circle of radius 1 centered at 0 travelled once counterclockwise. In part **(b)**, show that the integral exists, and show any contours and explain estimates needed to justify your method.

a. $\displaystyle\int_\gamma z^4 \sin(1/z)\, dz$ **b.** $\displaystyle\int_0^\infty \frac{dx}{x^4 + 16}$

Spring 2016 # 3. a. Find the image of the interior of the circle $C = \{z \in \mathbb{C} : |\, z - 2\,| = 2\}$ under the fractional linear transformation $w = f(z) = z/(2z - 8)$.

b. Find a fractional linear (Möbius) transformation mapping the region $D_1 = \{z \in \mathbb{C} : |\, z - 3\,| < 2\}$ onto the region $D_2 = \{w \in \mathbb{C} : \operatorname{Re} w < 0\}$

Spring 2016 # 4. Determine the number of zeros, counted according to their multiplicities, of the polynomial $p(z) = z^6 - 5z^4 + 10$ in each of the following sets.

a. $\{z \in \mathbb{C} : |\, z\,| < 1\}$ **b.** $\{z \in \mathbb{C} : 1 < |\, z\,| < 2\}$ **c.** $\{z \in \mathbb{C} : 2 < |\, z\,| < 3\}$

Spring 2016 # 5. a. (5 pts) Give a statement of Liouville's Theorem about analytic functions.

b. (15 pts) Suppose $f : \mathbb{C} \to \mathbb{C}$ is analytic on $\mathbb{C}$. Let $u(x, y) = \operatorname{Re}(f(x + iy)$ and $v(x, y) = \operatorname{Im}(f(x + iy)$, and suppose there is a positive real constant M with $|\, u(x, y)\,| \leq M$ for all $x + iy$ in $\mathbb{C}$. Show that there is a real constant c such that $v(x, y) = c$ for all $x + iy$ in $\mathbb{C}$. (Suggestion: Consider $|\, f(z) - (M + 1)\,|$. A picture might be helpful.)

Spring 2016 # 6. (Note: You do not need to know anything about Fourier series other than the definition here for this. It really is a complex analysis problem)

If $F(\vartheta)$ is a 2π-periodic function of ϑ, the Fourier coefficients of F are defined for integer n by $\hat{F}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(\vartheta) e^{-in\vartheta}\, d\vartheta$. Suppose that $r > 1$ and that $f(z)$ is a complex valued function analytic on the disk $D = \{z \in \mathbb{C} : |\, z\,| < r\}$. Let $F(\vartheta) = f(e^{i\vartheta})$.

a. Show that $\hat{F}(n) = 0$ for $n < 0$, and $\hat{F}(n) = f^{(n)}(0)/n!$ for $n \geq 0$.

b. Show that the series $\sum_{n=0}^\infty \hat{F}(n) e^{in\vartheta}$ converges to $F(\vartheta)$ for each real ϑ.

Spring 2016 # 7. Let $f(z) = z^2/(e^z - 1)$.

a. Find all the singularities of f in $\mathbb{C}$ and classify each as a removable singularity a pole, or an essential singularity. For poles, specify the order.

b. Evaluate $\int_\gamma f(z)\, dz$ for each of the following paths γ.

(i) the circle of radius 1 center 0 traveled once counterclockwise

(ii) the circle of radius 8 center 0 traveled once counterclockwise

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End of Exam