California State University – Los Angeles
Mathematics
Masters Degree Comprehensive Examination

Complex Analysis Spring 2020
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Do five of the following seven problems.
If you attempt more than 5, the best 5 will be used.

Please
(1) Write in a fairly soft pencil (number 2) (or in ink if you wish) so that your work will duplicate well. There should be a supply available.
(2) Write on one side of the paper only.
(3) Begin each problem on a new page.
(4) Assemble the problems you hand in in numerical order.

Exams are being graded anonymously, so put your name only where directed and follow any instructions concerning identification code numbers.

Notation: **C** denotes the set of complex numbers.
R denotes the set of real numbers.
Re(z) denotes the real part of the complex number z.
Im(z) denotes the imaginary part of the complex number z.
$|z|$ denotes the absolute value of the complex number z.
Log z denotes the principal branch of log z.
Arg z denotes the principal branch of arg z.
$D(z; r)$ denotes the open disk with center z and radius r.
A *domain* is an open connected subset of **C**.

<u>Miscellaneous facts</u>

$2\sin a \sin b = \cos(a-b) - \cos(a+b)$	$2\cos a \cos b = \cos(a-b) + \cos(a+b)$
$2\sin a \cos b = \sin(a+b) + \sin(a-b)$	$2\cos a \sin b = \sin(a+b) - \sin(a-b)$
$\sin(a+b) = \sin a \cos b + \cos a \sin b$	$\cos(a+b) = \cos a \cos b - \sin a \sin b$

Spring 2020 # 1. Determine which of the following functions $u(x, y)$ are harmonic. For each one that is harmonic, find a conjugate harmonic function $v(x, y)$ and express it as an analytic function $f = u + iv$ where $f(0) = 0$.

(a) $u(x, y) = 3x^2y + 2x^2 - y^3 - 2y^2$

(b) $u(x, y) = 2xy + 3xy^2 - 2y^3$

Spring 2020 # 2. Evaluate the following integrals.

(a) $\displaystyle\int_{|z+1|=3} \frac{z^2+2}{z^2+2z}\, dz$

(b) $\displaystyle\int_0^{\infty} \frac{x^2\cos(x)}{(1+x^2)^2}\, dx$

Spring 2020 # 3. Let $f(z) = \dfrac{1}{z(z^2+1)}$

(a) Find the Laurent series for $f(z)$ around $z_0 = 0$ and the annulus of convergence.

(b) Compute the residue of $f(z)$ at $z_0 = 0$.

(c) Find the Laurent series for $f(z)$ around $z_0 = i$ and the annulus of convergence.

(d) Compute the residue of $f(z)$ at $z_0 = i$.

Spring 2020 # 4. Let $A = \{z : \text{Im}(z) > 0\}$. For each of the following sets B determine if there exists a conformal map $\phi : A \to B$.

(a) $B = \mathbb{C} - \{z : \text{Im}(z) = 0\}$

(b) $B = \mathbb{C} - \{z : \text{Im}(z) = 0 \text{ and } |z| \geq 1\}$

Spring 2020 #5. Suppose that $u(x, y)$ is harmonic and bounded, prove that it must be constant. [Hint: Let $f(z) = f(x + iy) = u(x, y) + iv(x, y)$, where $v(x, y)$ is the harmonic conjugate of $u(x, y)$ and consider $e^{f(z)}$.

Spring 2020 # 6. Let $U = \{z \ : \ |z| < 1 \text{ or } |z| > 2\}$ and let $f : U \to \mathbb{C}$ be defined by

$$f(z) = \begin{cases} z & , \text{ if } |z| < 1 \\ z^2 & , \text{ if } |z| > 2 \end{cases}$$

Determine if there exists and entire function that agrees with f on U.

Spring 2020 # 7. Prove that the roots of $z^7 - 5z^3 + 12 = 0$ lie between the circles $|z| = 1$ and $|z| = 2$.