Circular Nim Games

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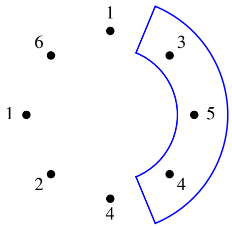
Circular Nim $\mathrm{CN}(n,k)$

- n stacks of tokens arranged in a circle
- Select k **consecutive** stacks and remove at least one token from at least one of the stacks
- Last player to move wins



$k = 1$ corresponds to regular Nim

Circular Nim $\mathrm{CN}(n,k)$

Question: For a given position, can we determine whether Player I or Player II has a winning strategy, that is, can make moves in such a way that s/he will win, no matter how the other player plays?

We will determine the set of losing positions, that is, all positions that result in a loss for the player playing from that position.

Combinatorial Games

Definition

An *impartial combinatorial game* has the following properties:

- each player has the **same moves** available at each point in the game (as opposed to chess, where there are white and black pieces).
- no randomness (dice, spinners) is involved, that is, each player has **complete information** about the game and the potential moves

Analyzing $\mathrm{CN}(n, k)$

Definition

A *position* in $\mathrm{CN}(n, k)$ is denoted by $\mathbf{p} = (p_1, p_2, \ldots, p_n)$, where $p_i \geq 0$ denotes the number of tokens in stack i. A position that arises from a move in the current position is called an *option*. The directed graph which has the positions as the nodes and an arrow between a position and its options is called the *game tree*.

We do not distinguish between a position and any of its rotations or reversals.

Options of position $(0, 1, 2)$ in $\mathrm{CN}(3, 2)$

$(\color{blue}{0}, \color{blue}{1}, 2) \quad \rightsquigarrow$

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$(0, 1, 2) \quad \rightsquigarrow$

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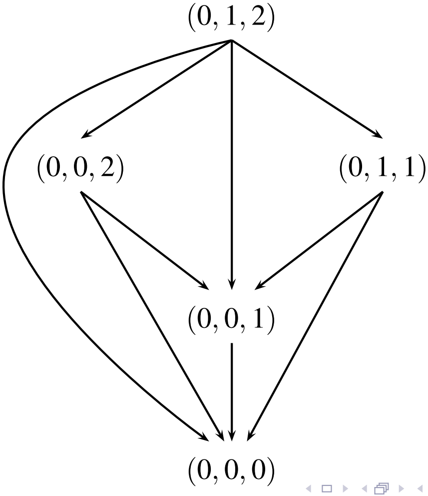
$$(0, \color{blue}{1}, \color{blue}{2}) \quad \rightsquigarrow \quad (0, 0, 2), (0, 0, 1), (0, 0, 0), (0, 1, 1), \color{red}{(0, 1, 0)}$$

$$(\color{blue}{0}, 1, \color{blue}{2}) \quad \rightsquigarrow \quad (0, 1, 1), (0, 1, 0)$$

Overall

$$(0, 1, 2) \quad \rightsquigarrow \quad (0, 0, 2), (0, 0, 1), (0, 0, 0), (0, 1, 1)$$

Game tree for CN(3, 2) position (0, 1, 2)



Impartial Games

Definition

A position is a $\mathcal{P}$ *position* for the player about to make a move if the $\mathcal{P}$revious player can force a win (that is, the player about to make a move is in a losing position). The position is a $\mathcal{N}$ *position* if the $\mathcal{N}$ext player (the player about to make a move) can force a win.

For impartial games, there are only two outcome classes for any position, namely **winning position** ($\mathcal{N}$ position) or **losing position** ($\mathcal{P}$ position). The set of losing positions is denoted by $\mathcal{L}$.

Recursive labeling

To find out whether Player I has a winning strategy, we label the nodes of the game tree **recursively** as follows:

- Leaves of the game tree are always losing ($\mathcal{P}$) positions.

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- The position has at least one option that is a losing ($\mathcal{P}$) position $\Rightarrow$ winning position and should be labeled $\mathcal{N}$
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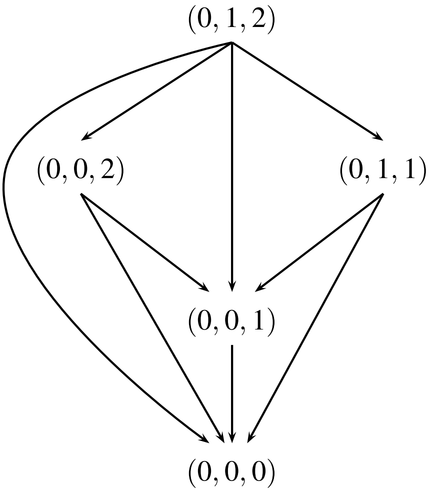
- The position has at least one option that is a losing ($\mathcal{P}$) position $\Rightarrow$ winning position and should be labeled $\mathcal{N}$
- All options of the position are winning ($\mathcal{N}$) positions $\Rightarrow$ losing position and should be labeled $\mathcal{P}$

The label of the starting position of the game then tells whether Player I ($\mathcal{N}$) or Player II ($\mathcal{P}$) has a winning strategy.

Labeling the game tree for $\mathrm{CN}(3, 2)$ position $(0, 1, 2)$



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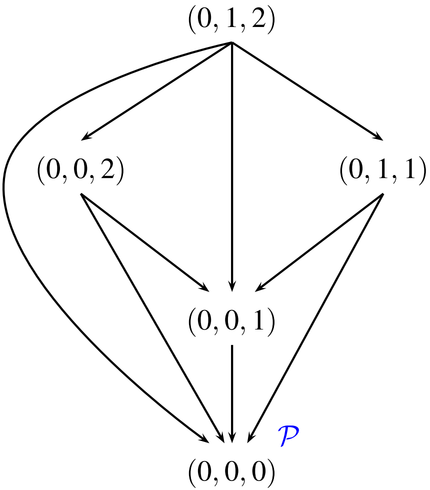

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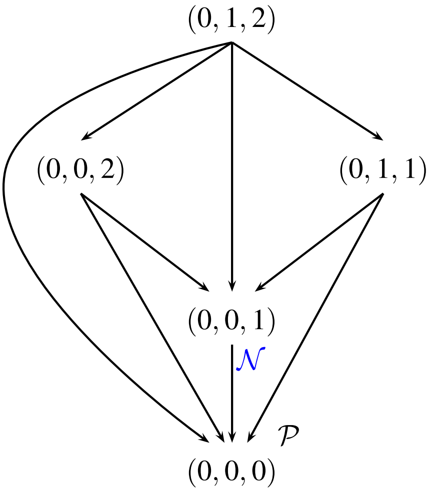

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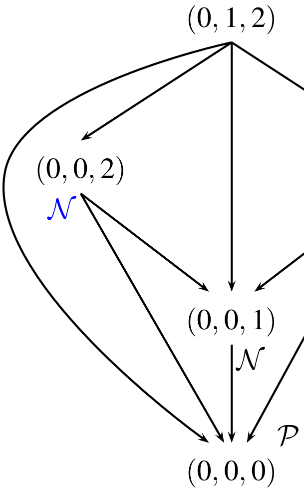

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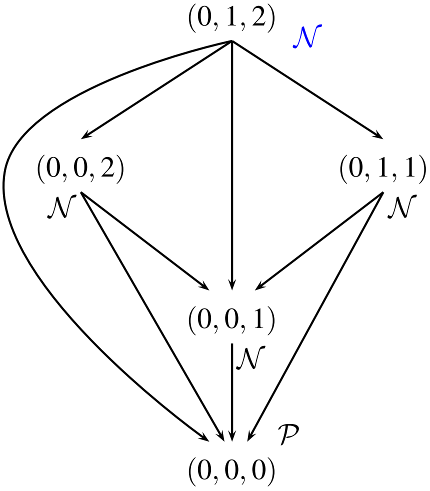

An important tool

Theorem

Suppose the positions of a finite impartial game can be partitioned into mutually exclusive sets A and B with the properties:

I. *every option of a position in A is in B;*
II. *every position in B has at least one option in A; and*
III. *the final positions are in A.*

Then $A = \mathcal{L}$ *and* $B = \mathcal{W}$.

Proof strategy

- Obtain a candidate set S for the set of losing positions $\mathcal{L}$
- Show that any move from a position $\mathbf{p} \in S$ leads to a position $\mathbf{p}' \notin S$ (I)
- Show that for every position $\mathbf{p} \notin S$, there is a move that leads to a position $\mathbf{p}' \in S$ (II)

Note that the only final position is $(0, 0, \ldots, 0)$, and it is easy to see that (III) is satisfied in all cases.

Digital sum

Definition

The *digital sum* $a \oplus b \oplus \cdots \oplus k$ of of integers $a, b, \ldots, k$ is obtained by translating the values into their binary representation and then adding them without carry-over.

Note that $a \oplus a = 0$.

Example

The digital sum $12 \oplus 13 \oplus 7$ equals 6:

$$\begin{array}{c|cccc} 12 & 1 & 1 & 0 & 0 \\ 13 & 1 & 1 & 0 & 1 \\ 7 & & 1 & 1 & 1 \\ \hline & 0 & 1 & 1 & 0 \end{array}$$

The easy cases

Theorem

(1) *The game* $\mathrm{CN}(n,1)$ *reduces to Nim, for which the set of losing positions is given by*
$\mathcal{L} = \{(p_1, p_2, \ldots, p_n) \mid p_1 \oplus p_2 \oplus \cdots \oplus p_n = 0\}$.

(2) *The game* $\mathrm{CN}(n,n)$ *has a single losing position, namely*
$\mathcal{L} = \{(0, 0, \ldots, 0)\}$.

(3) *The game* $\mathrm{CN}(n,n-1)$ *has losing positions*
$\mathcal{L} =$

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$\mathcal{L} = \{(a, a, \ldots, a) \mid a \geq 0\}$.

This covers the games for $n = 1, 2, 3$. For $n = 4$, the only one game to consider is $\mathrm{CN}(4,2)$.

Result for $\mathrm{CN}(4,2)$

Theorem

For the game $\mathrm{CN}(4,2)$*, the set of losing positions is*
$\mathcal{L} = \{(a,b,a,b) \mid a,b \geq 0\}$.

Result for $\mathrm{CN}(4, 2)$

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Proof.

Let $S = \{(a, b, a, b)\}$ and $\mathbf{p} \in S$. Playing on any stack results in a different value in its diagonal opposite stack $\Rightarrow \mathbf{p}' \notin S$.

$\square$

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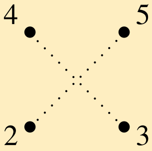

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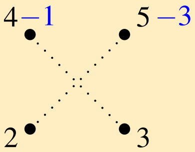

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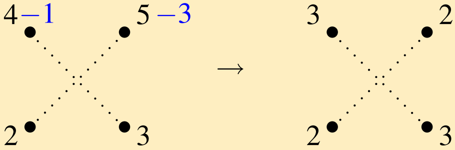
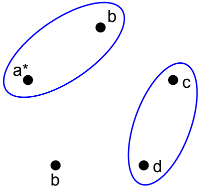


$\square$

Result for $\mathrm{CN}(5,2)$

Theorem (Dufour; Ehrenborg & Steingrímsson)

The game $\mathrm{CN}(5,2)$ *has losing positions*
$\mathcal{L} = \{(a^*, b, c, d, b) \mid a^* + b = c + d, a^* = \max(\mathbf{p})\}$.

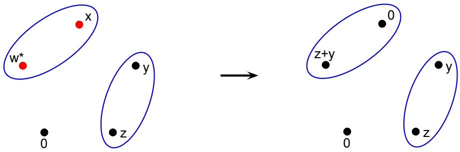


Note that b has to be $\min(\mathbf{p})$.

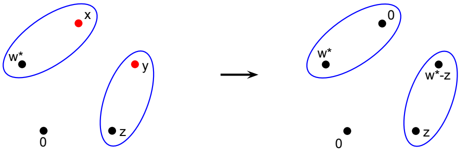
Result for $\mathrm{CN}(5,2)$

To show part (II), we can assume that $\min(\mathbf{p}) = 0$. Two cases:
(i) $\max(\mathbf{p}) = w^*$ and $\min(\mathbf{p})$ adjacent, $\mathbf{p} = (0, w^*, x, y, z)$

- $w^* \geq z + y$:



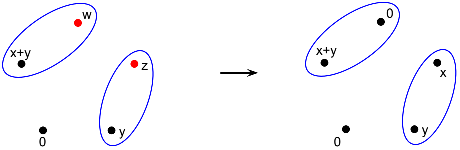
- $w^* < z + y$:



Result for $\mathrm{CN}(5,2)$

(ii) $\max(\mathbf{p})$ and $\min(\mathbf{p})$ separated by one stack, $\mathbf{p} = (0, x+y, w, z, y)$, $\max(\mathbf{p}) \in \{w, z\}$

- $z \geq x$:



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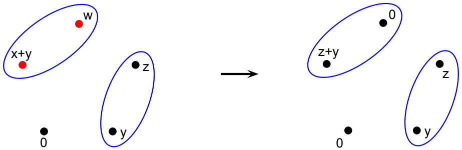
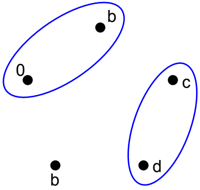


Result for $\mathrm{CN}(5,3)$

Theorem (Ehrenborg & Steingrímsson)

The game $\mathrm{CN}(5,3)$ *has losing positions*
$\mathcal{L} = \{(0,b,c,d,b) \mid b = c + d\}$.



Note that b has to be $\max(\mathbf{p})$. Proof similar to $\mathrm{CN}(5,2)$ with more cases to be considered.

The big question

How do we find $\mathcal{L}$????

Mex

Definition

The *minimum excluded value* or *mex* of a set of non-negative integers is the least non-negative integer which does not occur in the set. It is denoted by $\text{mex}\{a, b, c, \ldots, k\}$.

Example

$$\text{mex}\{1, 4, 5, 7\} =$$
$$\text{mex}\{0, 1, 2, 6\} =$$

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$$\text{mex}\{1, 4, 5, 7\} = 0$$
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Example

$$\text{mex}\{1, 4, 5, 7\} = 0$$
$$\text{mex}\{0, 1, 2, 6\} = 3$$

The Grundy Function

Definition

The Grundy function $\mathcal{G}(\mathbf{p})$ of a position $\mathbf{p}$ is defined recursively as follows:

- $\mathcal{G}(\mathbf{p}) = 0$ for any final position $\mathbf{p}$.
- $\mathcal{G}(\mathbf{p}) = \text{mex}\{\mathcal{G}(\mathbf{q}) | \mathbf{q} \text{ is an option of } \mathbf{p}\}$.

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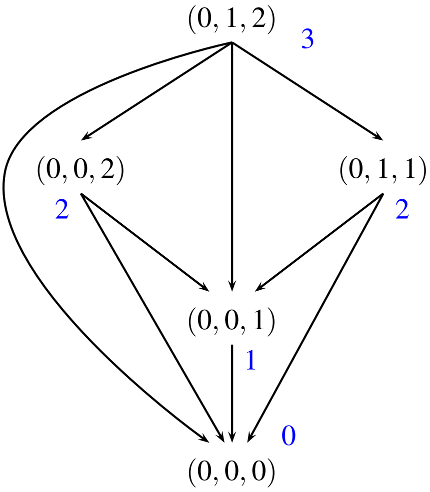
Theorem

For a finite impartial game, $\mathbf{p}$ *belongs to class* $\mathcal{P}$ *if and only if* $\mathcal{G}(\mathbf{p}) = 0$.

Recursive computation of Grundy function



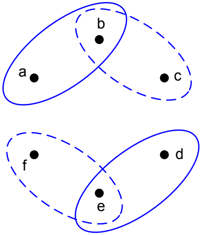
Finding candidate set for $\mathcal{L}$

- Write program that computes options for a given position and then recursively computes Grundy function for each position
- Filter out those positions that have Grundy value zero
- CREATIVITY - find pattern
- Write program that computes values to check your pattern
- If pattern holds for large enough number of examples, try to prove it!

Result for CN$(6,3)$

Theorem

For the game CN$(6,3)$, *the set of losing positions is given by*
$\mathcal{L} = \{(a,b,c,d,e,f) | a+b = d+e, b+c = e+f\}$.



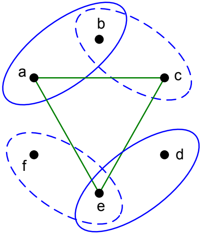
Note that also $c+d = f+a$.

Result for $\mathrm{CN}(6,4)$

Theorem

For the game $\mathrm{CN}(6,4)$, *the set of losing positions is given by*

$$\mathcal{L} = \{(a,b,c,d,e,f) | a+b = d+e, b+c = e+f, a \oplus c \oplus e = 0, \quad a = min(\mathbf{p})\}.$$



Note that also $c + d = f + a$.

Proof of $\mathcal{L}_{\text{CN}(6,4)}$ uses two lemmas:

Lemma

If the position $\mathbf{p} = (a, b, c, d, e, f) \in \mathcal{L}_{\text{CN}(6,4)}$ *has a minimal value in each of the two triples* (a, c, d) *and* (b, d, f)*, then* $\mathbf{p} = (a, b, c, a, b, c)$.

Lemma

For any set of positive integers $x_1, x_2, \ldots, x_n$ *there exists an index* i *and a value* x_i' *such that* $0 \leq x_i' \leq x_i$ *and*

$$x_1 \oplus \cdots \oplus x_{i-1} \oplus x_i' \oplus x_{i+1} \oplus \cdots \oplus x_n = 0.$$

Result for $\mathrm{CN}(6,2)$

?????

- Difficult case to prove - we need ALL Grundy values for a special substructure
- Same substructure occurs in all $\mathrm{CN}(n,2)$ games for $n \geq 6$
- Structure also occurs in other games such as $\mathrm{CN}(9,3)$

Conjecture for $\mathrm{CN}(2m, m)$

$$\mathcal{L}_{\mathrm{CN}(4,2)} = \{(a, b, c, d) \mid a + b = c + d \wedge b + c = a + d\}$$

$$\mathcal{L}_{\mathrm{CN}(6,3)} = \{(a, b, c, d, e, f) | a + b = d + e \wedge b + c = e + f\}$$

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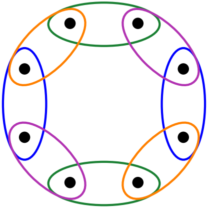
Conjecture:
Sums of pairs that are diagonally across are the same



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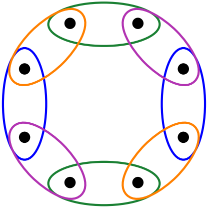
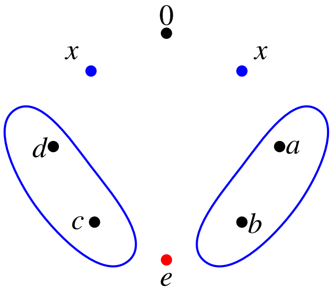
$$\mathcal{L}_{\mathrm{CN}(6,3)} = \{(a,b,c,d,e,f) | a+b = d+e \wedge b+c = e+f\}$$

Conjecture:
Sums of pairs that are diagonally across are the same **NO**



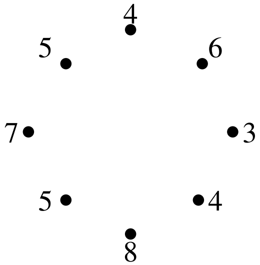
$n \geq 7$

- We have some partial results/conjectures for $n = 7, 8, 9$.
- Specifically, $\mathcal{L}_{\text{CN}(8,6)} = \{(0, x, a, b, e, c, d, x) \mid a + b = c + d = x, e = \min\{x, a + d\}\}$.



Example for $CN(8,6)$

Can you find a move that results in a losing position?



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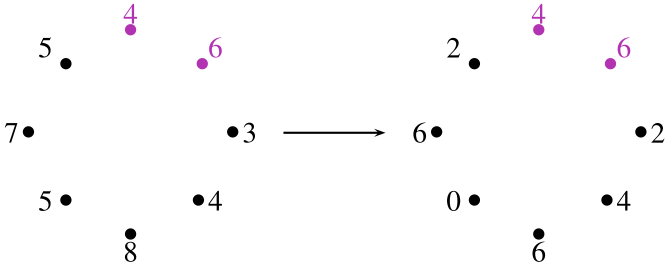

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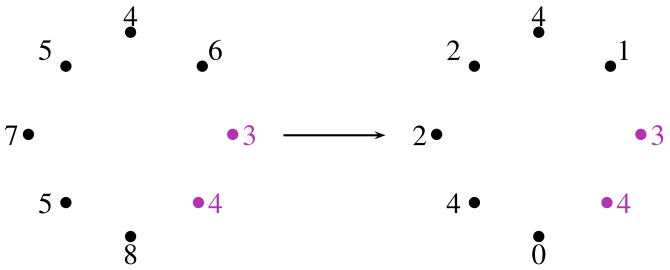

Variations of Circular Nim

- Select a fixed number a from at least one of the stacks

Variations of Circular Nim

- Select a fixed number a from at least one of the stacks
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- Select a fixed number a from at least one of the stacks
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- Select at least one token from each of the k heaps
- Select at least a tokens from each of the k heaps

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Note that there is a different dynamic when the requirement is to select from each stack, as a zero stack now splits the position into separate positions with smaller n and symmetries disappear.

Variations of Circular Nim

- Select a fixed number a from at least one of the stacks
- Select a fixed number a from each of the heaps
- Select at least one token from each of the k heaps
- Select at least a tokens from each of the k heaps
- Select a total of at least a tokens from the k stacks

Note that there is a different dynamic when the requirement is to select from each stack, as a zero stack now splits the position into separate positions with smaller n and symmetries disappear.

Variations of Circular Nim

- Select a fixed number a from at least one of the stacks
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- Select at least one token from each of the k heaps
- Select at least a tokens from each of the k heaps
- Select a total of at least a tokens from the k stacks
- Select a total of exactly a tokens from the k stacks

Note that there is a different dynamic when the requirement is to select from each stack, as a zero stack now splits the position into separate positions with smaller n and symmetries disappear.

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Thank You!

References and Further Reading

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