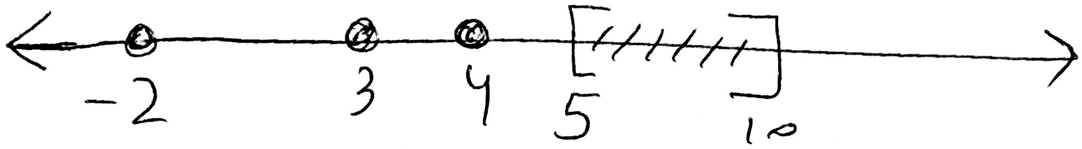
Test 1, Math 4650, Fall 2017 **Name:** Solutions

1. [15 points - 5 each] Find the supremum and infimum of each set if they exist.
First draw a picture of the set or list several elements of the set to get an idea of what's going on.

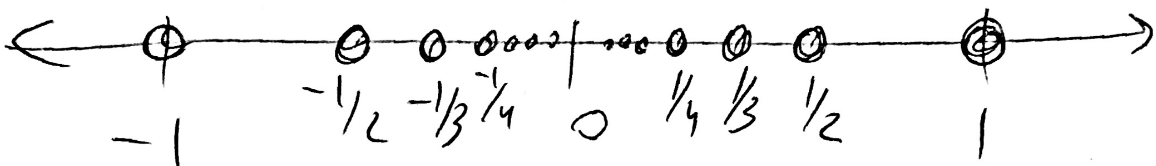
(a) $X = [5, 10] \cup \{3, -2, 4\}$



$\inf(X) = -2$ $\sup(X) = 10$

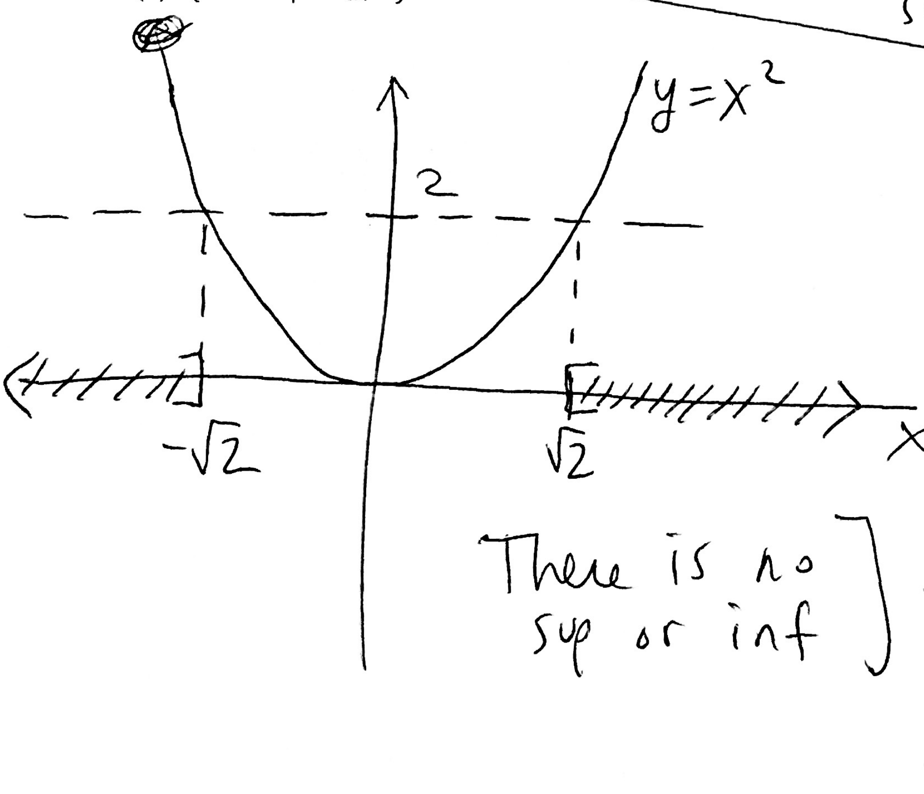
(b) $Y = \left\{ \frac{1}{n} \;\middle|\; n = 1, -1, 2, -2, 3, -3, 4, -4, \ldots \right\}$

$= \left\{ 1, -1, \frac{1}{2}, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{4}, \ldots \right\}$



$\inf(Y) = -1$
$\sup(Y) = 1$

(c) $\{x \in \mathbb{R} \mid x^2 \geq 2\}$



There is no sup or inf

shaded part is the set

2. [10 points] Prove that

$$\lim_{n\to\infty} \frac{-n^2-3}{1+2n^2} = \frac{-1}{2}$$

To get any credit you must use the definition of limit as we did in class and in hw.

Let $\varepsilon > 0$.

$\boxed{\underline{\text{Goal}}\text{: Find } N \text{ where if } n \geq N \text{ then } \left|\frac{-n^2-3}{1+2n^2} - \left(-\frac{1}{2}\right)\right| < \varepsilon.}$

Note that

$$\left|\frac{-n^2-3}{1+2n^2} + \frac{1}{2}\right| = \left|\frac{-2n^2-6+1+2n^2}{2(1+2n^2)}\right| = \left|\frac{-5}{2+4n^2}\right| = \frac{5}{2+4n^2}$$

And $\frac{5}{2+4n^2} < \varepsilon$ iff $\frac{5}{\varepsilon} < 2+4n^2$ iff

$\frac{\frac{5}{\varepsilon}-2}{4} < n^2$ iff $\sqrt{\frac{\frac{5}{\varepsilon}-2}{4}} < n$.

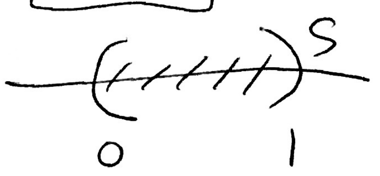
Let $N > \sqrt{\frac{\frac{5}{\varepsilon}-2}{4}}$, Then from above,

if $n \geq N$ then $\left|\frac{-n^2-3}{1+2n^2} - \left(-\frac{1}{2}\right)\right| < \varepsilon$.

3. [10 points - 5 each] True or False. Directions: If True, write "True" and give a short proof. You must prove it to get credit. If False, write "False" and give an explicit example to illustrate where the statement fails.

(a) Suppose that S is a non-empty subset of the real numbers and that S is bounded from above. Let b be the supremum of S. Then $b \in S$.

$\boxed{\text{False}}$ Let $S = (0, 1)$



Then $1 = \sup(S)$ but $1 \notin S$

(b) Suppose that S is a non-empty subset of the real numbers that is bounded from above. If x is an upper bound for S and $x \in S$, then x is the supremum of S.

$\boxed{\text{True}}$

pf: ① We are given that x is an upper bound for S.

② Suppose b is another upper bound for S. Then $s \leq b$ for all $s \in S$. Since $x \in S$, then $x \leq b$. So x is the least upper bound for S,

By ① + ②, $x = \sup(S)$,

4. [10 points] Suppose that (a_n) and (b_n) are sequences of real numbers. Suppose further that $\lim_{n\to\infty} a_n = 0$ and $\lim_{n\to\infty} b_n = B$. Let α be a real number with $\alpha \neq 0$. Prove that $\lim_{n\to\infty}(-3a_n + \alpha b_n + 5) = \alpha B + 5$.

To receive any credit for this problem you must use the ϵ-N-definition of the limit to prove this result. No using theorems that we proved in class or hw like $\lim(a_n+b_n) = \lim a_n + \lim b_n$. That's not allowed on this problem.

Let $\varepsilon > 0$.
We want to find N where if $n \geq N$ then

$$|(-3a_n + \alpha b_n + 5) - (\alpha B + 5)| < \varepsilon.$$

Note that

$$|(-3a_n + \alpha b_n + 5) - (\alpha B + 5)| = |(-3a_n - 0) + \alpha(b_n - B)|$$
$$\leq |-3a_n - 0| + |\alpha||b_n - B|$$
$$= 3|a_n - 0| + |\alpha||b_n - B|$$

Since $\lim_{n\to\infty} a_n = 0$, there exists N_1 where if $n \geq N_1$ then $|-3a_n - 0| < \frac{\varepsilon}{6}$.
Since $\lim_{n\to\infty} b_n = B$, there exists N_2 where if $n \geq N_2$ then $|b_n - B| < \frac{\varepsilon}{2|\alpha|}$.

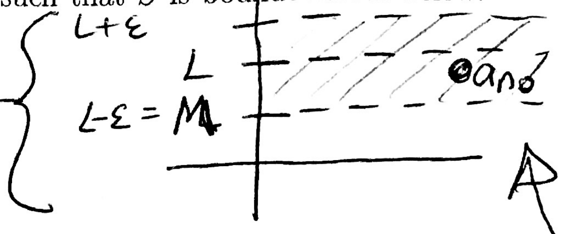
Let $N = \max\{N_1, N_2\}$. Then if $n \geq N$
then

$$|(-3a_n - \alpha b_n + 5) - (\alpha B + 5)| \leq 3|a_n - 0| + |\alpha||b_n - B|$$
$$< 3 \cdot \frac{\varepsilon}{6} + |\alpha| \cdot \frac{\varepsilon}{2|\alpha|} = \varepsilon,$$

5. [10 points] PICK <u>ONE</u> PROBLEM BELOW. ONLY CHOOSE ONE. IF YOU DO BOTH THEN I WILL GRADE (A).

A) Suppose that (a_n) is a convergent sequence. Suppose that there exists $M > 0$ such that $a_n \leq M$ for all n. Prove that if $\lim_{n\to\infty} a_n = L$ then $L \leq M$. (You must prove this one using the definition of limit. No theorems from class or hw.)

B) Let S be a non-empty subset of the real numbers such that S is bounded from below. Prove that $\inf(S) = -\sup\{-s \mid s \in S\}$.



A) Suppose that $L > M$.

Let $\varepsilon = L - M$.

Since $\lim_{n\to\infty} a_n = L$, there exists N where if $n \geq N$ then $|a_n - L| < \varepsilon$.

Pick some $n_0 \geq N$. Then $|a_{n_0} - L| < \varepsilon$.

So, $-\varepsilon < a_{n_0} - L < \varepsilon$.

So, $-(L-M) < a_{n_0} - L < L - M$.

(Add L through eqn)

So, $M < a_{n_0} < 2L - M$

But then $M < a_{n_0}$. which is a contradiction

5. [10 points] PICK ONE PROBLEM BELOW. ONLY CHOOSE ONE. IF YOU DO BOTH THEN I WILL GRADE (A).

A) Suppose that (a_n) is a convergent sequence. Suppose that there exists $M > 0$ such that $a_n \leq M$ for all n. Prove that if $\lim_{n\to\infty} a_n = L$ then $L \leq M$. (You must prove this one using the definition of limit. No theorems from class or hw.)

B) Let S be a non-empty subset of the real numbers such that S is bounded from below. Prove that $\inf(S) = -\sup\{-s \mid s \in S\}$.

B) Let $x = \inf(S)$.
(We know x exists since S is bounded from below).
Then $x \leq s$ for all $s \in S$.
So, $-x \geq -s$ for all $s \in S$.
So, $-x$ is an upper bound for $\{-s \mid s \in S\}$.

So, $\sup\{-s \mid s \in S\}$ exists, since the set has an upper bound.
Let $y = \sup\{-s \mid s \in S\}$.
Then since y is the least upper bound for $\{-s \mid s \in S\}$ and $-x$ is an upper bound of $\{-s \mid s \in S\}$ we know that $y \leq -x$.

So, $\boxed{-y \geq x.}$ $(*)$
Also, $y \geq -s$ $\forall s \in S$ since $y = \sup\{-s \mid s \in S\}$.
So, $-y \leq s$ $\forall s \in S$.
So, $-y$ is a lower bound for S.
Since x is the greatest lower bound (inf) of S, we have $\boxed{-y \leq x}$ $(**)$ Combining $(*)$ and $(**)$ gives $x = -y$.
Thus, $\inf(S) = -\sup\{-s \mid s \in S\}$.