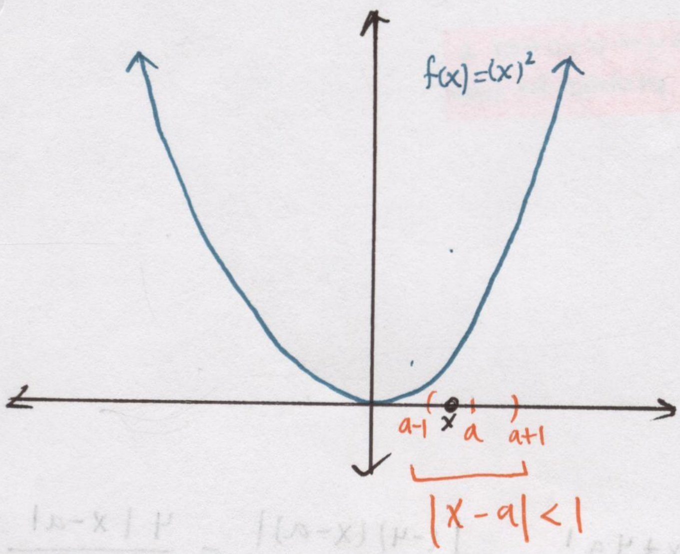
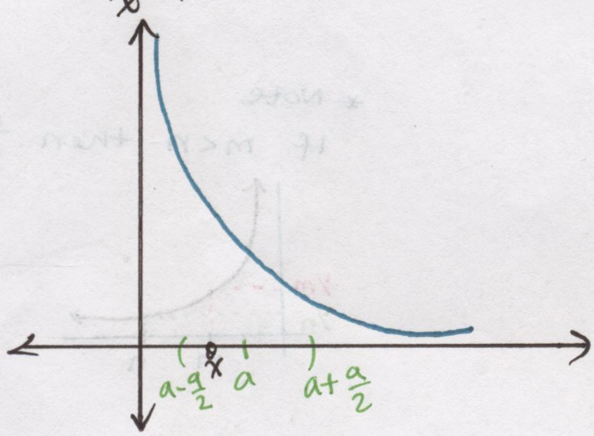
$f(x) = x^2$ is continuous for all a

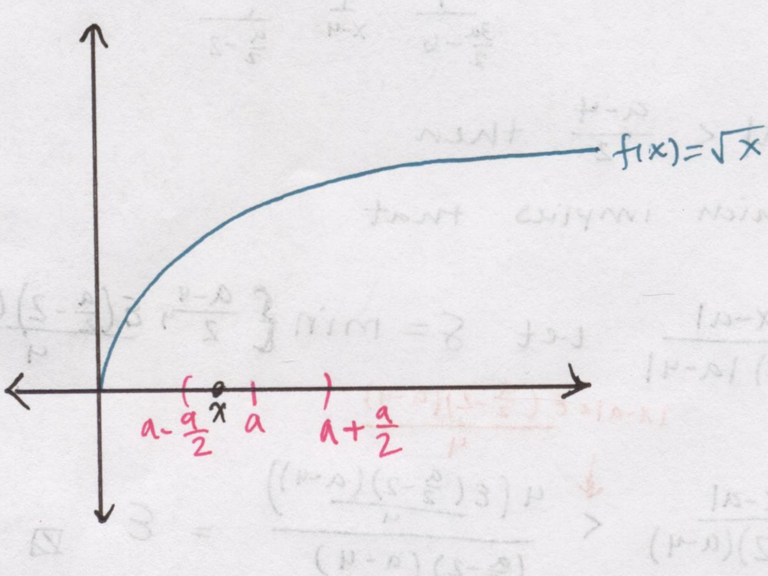
Assume $\delta \leq 1$



$f(x) = \frac{1}{x}$, $a > 0$. assume $\delta \leq \frac{a}{2}$

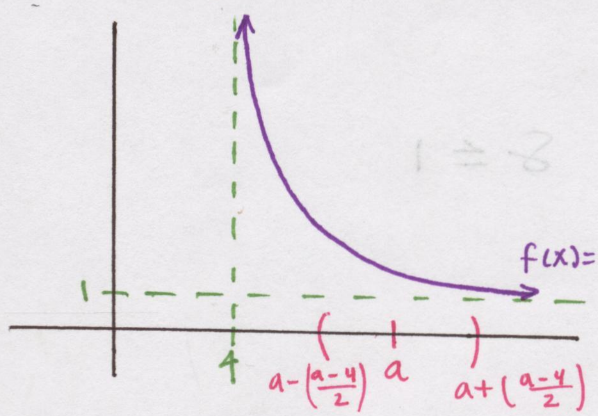


$f(x) = \sqrt{x}$. Assume $\delta \leq \frac{a}{2}$



Example: Let $f(x) = \frac{x}{x-4}$

Let $a > 4$, show f is continuous at a



proof: Let $\varepsilon > 0$,

Note that,

$$\left|\frac{x}{x-4} - \frac{a}{a-4}\right| = \left|\frac{x(a-4) - a(x-4)}{(x-4)(a-4)}\right| = \frac{|-4x+4a|}{|x-4||a-4|} = \frac{|(-4)(x-a)|}{|x-4||a-4|} = \frac{4|x-a|}{|x-4||a-4|}$$

Suppose $\delta \le \frac{a-4}{2}$

Let $x \in \mathbb{R}$ with $|x-a| < \frac{a-4}{2}$, $x \neq 4$

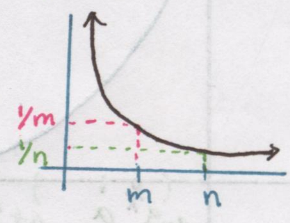
Thus, $\frac{a}{2} + 2 < x < \frac{3a}{2} - 2$] subtract 4

Thus, $\frac{a}{2} - 2 < x - 4 < \frac{3a}{2} - 6$] flip*

Thus, $\frac{1}{\frac{a}{2}-2} > \frac{1}{x-4} > \frac{1}{\frac{3a}{2}-6}$

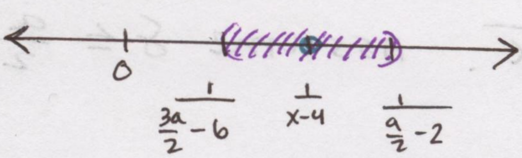
* Note

If $m < n$ then $\frac{1}{m} > \frac{1}{n}$



We have $a > 4$ so $\frac{3a}{2} > 6$

Thus, $\frac{3a}{2} - 6 > 0$



Thus, if $x = 4$ and $|x-a| < \frac{a-4}{2}$ then

$\left|\frac{1}{x-4}\right| = \frac{1}{|x-4|} < \frac{1}{\frac{a}{2}-2}$ which implies that

$$\left|\frac{x}{x-4} - \frac{a}{a-4}\right| = \frac{4|x-a|}{|x-4||a-4|} < \frac{4|x-a|}{(\frac{a}{2}-2)|a-4|}$$

Let $\delta = \min\left\{\frac{a-4}{2}, \frac{\varepsilon(\frac{a}{2}-2)(a-4)}{4}\right\}$

If $x = 4$ and $|x-a| < \delta$ then,

$$\left|\frac{x}{x-4} - \frac{a}{a-4}\right| = \frac{4|x-a|}{|x-4||a-4|} \underset{|x-a| < \frac{a-4}{2}}{<} \frac{4|x-a|}{(\frac{a}{2}-2)(a-4)} \overset{|x-a| < \frac{\varepsilon(\frac{a}{2}-2)(a-4)}{4}}{<} \frac{4\left(\frac{\varepsilon(\frac{a}{2}-2)(a-4)}{4}\right)}{(\frac{a}{2}-2)(a-4)} = \varepsilon \quad \boxtimes$$

Compactness continued...

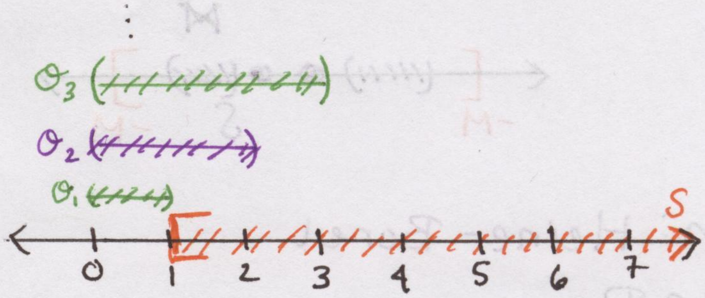
Ex: Let $S = [1, \infty)$

$X = \{ \mathcal{O}_n = (0,n) \mid n \in \mathbb{N} \} = \{ (0,1), (0,2), (0,3), \dots \}$

Is X a cover of S?

$S \subseteq \bigcup_n \mathcal{O}_n = (0, \infty)$

So, X covers S.



Does there exist $X' \subseteq X$ where X' is a finite subcover?

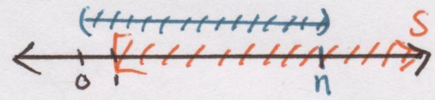
That is, X' is finite and still covers S.

NO! suppose $X' = \{ (0,n_1), (0,n_2), \dots, (0, n_k) \}$

Let $n = \max\{n_1, n_2, \dots, n_k\}$

Then $\bigcup_{\mathcal{O} \in X'} \mathcal{O} = \bigcup_{i=1}^{k} (0, n_i) = (0, n)$

This does not cover S!



Therefore, X is a cover of S with no finite subcover.

Def: Let $S \subseteq \mathbb{R}$

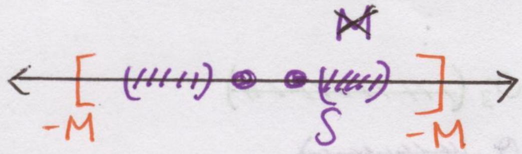
we say that S is **compact** if every open cover of S contains a finite subcover.

Ex: $S = [1, \infty)$ is not compact since $X = \{ (0,n) \mid n \in \mathbb{N} \}$ is an open cover of S with no finite subcover.

Def: let $S \subseteq \mathbb{R}$

We say that S is bounded if there exists $M > 0$

- where $|x| < M \;\forall\, x \in S$
- that is $S \subseteq [-M, M]$



Theorem: Heine-Borel

Let $S \subseteq \mathbb{R}$

S is compact iff S is closed and bounded.