Heine-Borel

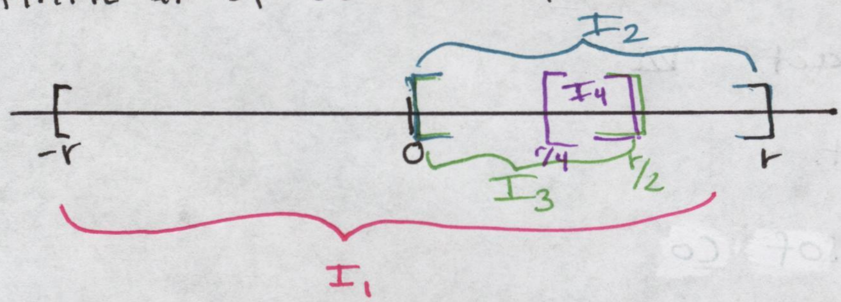
Let $K \subseteq \mathbb{R}$, K is compact iff K is closed and bounded

proof:

($\Leftarrow$) Suppose K is closed and bounded

summary of last time:

- Let $G = \{G_\alpha\}$ be an open cover of K
- Suppose K is not covered in a finite # of sets from G. } proof by contradiction
- created a sequence of closed intervals $\ldots \subseteq I_3 \subseteq I_2 \subseteq I_1 = [-r, r]$ where $K \subseteq [r, r]$ and $K \cap I_i \neq \emptyset$ and $K \cap I_i$ is not covered by a finite # of elements from G.



- let $I_n = [a_n, b_n]$

  Let $f = \sup\{a_k \mid k \geq 1\}$

  we showed $f = \inf\{b_k \mid k \geq 1$

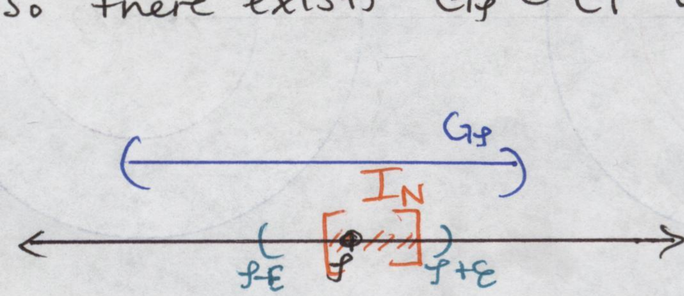
  we showed f is a limit point of K

  we showed $f \in \bigcap_{n=1}^{\infty} I_n$

continuation from last time:

-since K is closed, K contains all its limit points.

So, $f \in K$. So there exists $G_f \in G$ where $f \in G_f$



- since G_ξ is an open set there exists $\varepsilon > 0$ where $(\xi-\varepsilon, \xi+\varepsilon) \subseteq G_\xi$

Note that the length of $I_n = \frac{r}{2^{n-2}}$

since r is fixed, $\frac{r}{2^{n-2}} \to 0$ as $n \to \infty$

so there must exist $N > 0$ where $\frac{r}{2^{N-2}} < \varepsilon$

since $\xi \in I_N$ we must have $I_N \subseteq (\xi-\varepsilon, \xi+\varepsilon)$

But then $K \cap I_N$ is covered by G_ξ.

CONTRADICTION!

Thus, K can't be covered by a finite # of sets from G.

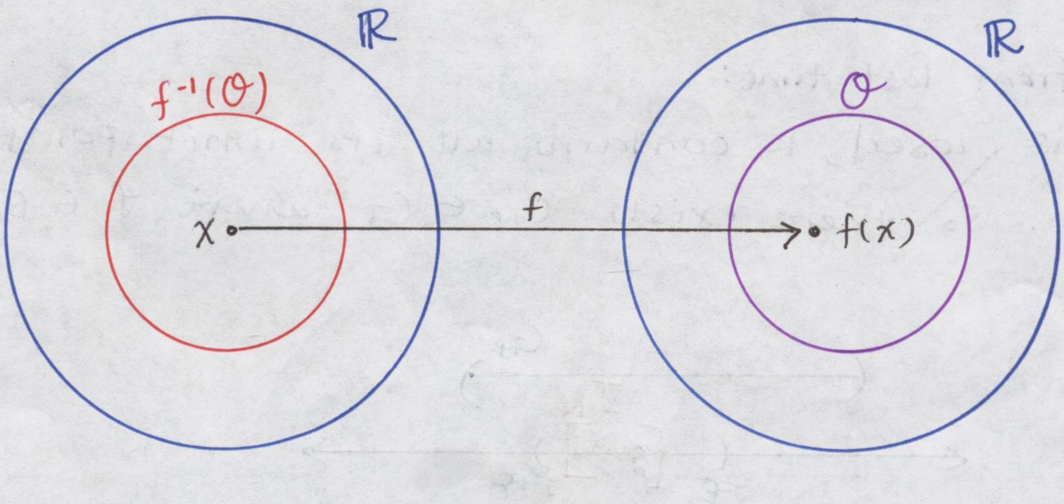
So K is compact. ▨

($\Rightarrow$) In Handout.

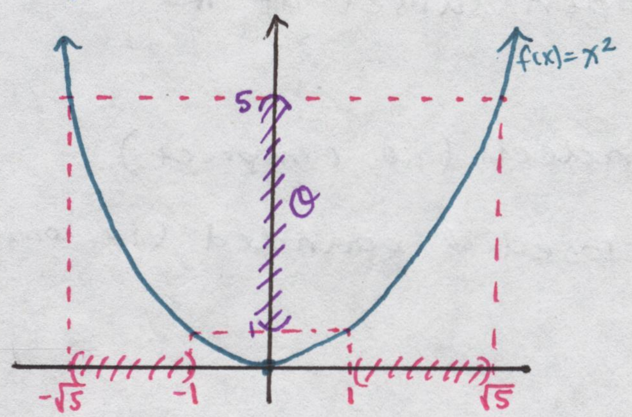
3 More Theorems

Theorem: Let $f: D \to \mathbb{R}$ where D is an open set.

Let $\mathcal{O} \subseteq \mathbb{R}$. If f is continuous on D, then $f^{-1}(\mathcal{O}) = \{x \in D \mid f(x) \in \mathcal{O}\}$ is open.



Example: $f(x) = x^2 \quad \mathcal{O} = (1, 5)$



$f(x) = x^2$ is continuous on all of $\mathbb{R}$.

$\mathcal{O} = (1, 5)$ is open

$f^{-1}(\mathcal{O}) = (-\sqrt{5}, -1) \cup (1, \sqrt{5})$ is open

proof of thm:

Let $\mathcal{O} \subseteq \mathbb{R}$ be open.

We want to show that $f^{-1}(\mathcal{O})$ is open

Let $a \in f^{-1}(\mathcal{O})$

so $f(a) \in \mathcal{O}$

since $\mathcal{O}$ is open, there exists $\varepsilon > 0$ where $(f(a)-\varepsilon, f(a)+\varepsilon) \subseteq \mathcal{O}$

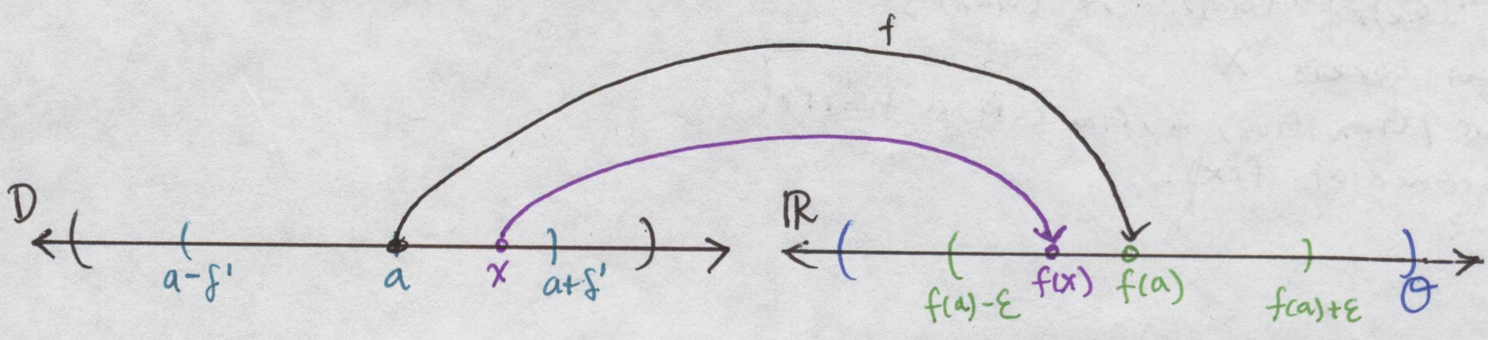
since f is continuous, there exists $\delta > 0$ where if $x \in D$ and $|x-a| < \delta$ then $|f(x) - f(a)| < \varepsilon$.

Since D is open, there exists $\delta' \leq \delta$ where $(a-\delta', a+\delta') \subseteq D$.

So if $|x-a| < \delta' \leq \delta$, then $x \in D$ and $|f(x) - f(a)| < \varepsilon$

so if $\underbrace{x \in (a-\delta', a+\delta')}_{\text{same as: } |x-a|<\delta'}$ then $\underbrace{f(x) \in (f(a)-\varepsilon, f(a)+\varepsilon)}_{\text{same as: } |f(x)-f(a)|<\varepsilon} \subseteq \mathcal{O}$

so, $(a-\delta', a+\delta') \subseteq f'(\mathcal{O})$. That is $f'(\mathcal{O})$ is open $\blacksquare$



Theorem:

Let $f: D \to \mathbb{R}$, where D is an open subset of $\mathbb{R}$

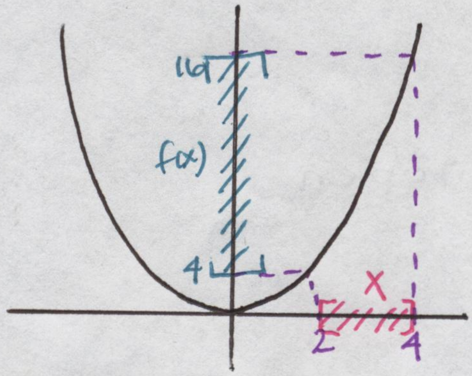
Let f be continuous on D.

If $X \subseteq D$ is closed and bounded (i.e. compact),

then $f(X) = \{f(a) \mid a \in X\}$ is closed & bounded (i.e. compact

Example: $f(x) = x^2$

$X = [2,4]$



f is continuous on all of $\mathbb{R}$

$X = [2,4]$ is compact

$f(X) = [4,16]$ is compact

proof of Theorem: Suppose $X \subseteq D$ is compact

consider $f(X) = \{f(a) \mid a \in X\}$. Let's show $f(X)$ is compact

Let $G = \{G_\alpha\}$ be an open cover of $f(X)$.

So every $G_\alpha \in G$ is open and $f(X) \subseteq \bigcup_\alpha G_\alpha$

Let $G = \{f^{-1}(G_\alpha)\}$ Then G is an open cover of X since

(1) $f^{-1}(G_\alpha)$ is open $\forall \alpha$ since G_α is open & f is continuous

(2) If $a \in X$, then $f(a) \in f(X)$. So $f(a) \in G_\alpha$ for some α so $a \in f^{-1}(G_\alpha$

- Since X is compact there is a finite subcover

$\{f^{-1}(G_{\alpha_1}), f^{-1}(G_{\alpha_2}), \dots, f^{-1}(G_{\alpha_n})\}$

that covers X.

- Then $\{G_{\alpha_1}, G_{\alpha_2}, \dots, G_{\alpha_n}\}$ is a finite subcover of $f(x)$.

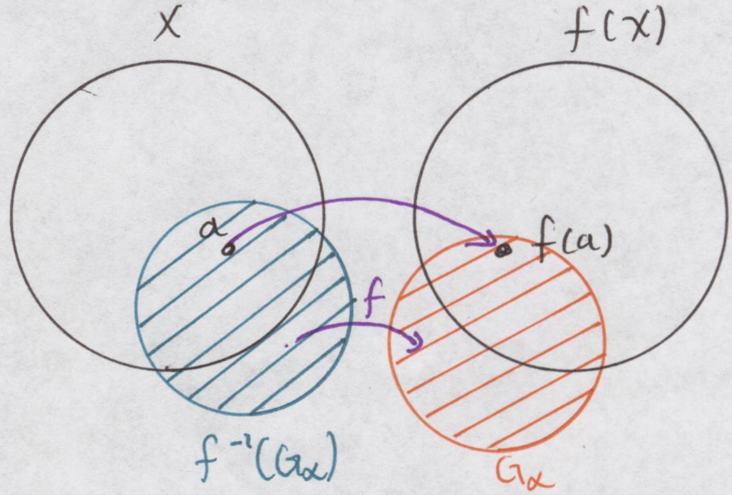
why?

P.3 11/26
why?

If $f(a) \in f(X)$ for some $a \in X$

then $a \in f^{-1}(G_{\alpha_i})$ for some i and so

$f(a) \in G_{\alpha_i}$ ☒



since X is compact

there is a finite subcover

$\{f^{-1}(G_{\alpha_1}), f^{-1}(G_{\alpha_2}), \ldots, f^{-1}(G_{\alpha_n})\}$

that covers X.

Corollary: Suppose $f: D \to \mathbb{R}$ where $D \subseteq \mathbb{R}$ is open.

Suppose f is continuous on D.

Let $X \subseteq D$ where X is compact (closed & bounded)

Then there exists $a, b \in X$ where

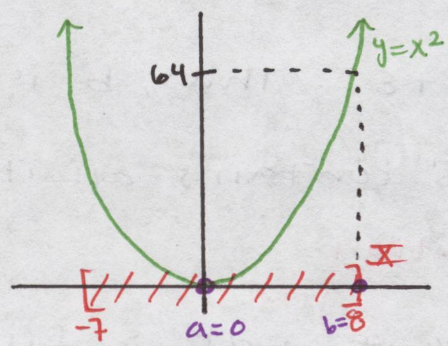
- $f(a) \leq f(x)$ for all $x \in X$ ← $f(a)$ is a minimum of f on X

and
- $f(x) \leq f(b)$ for all $x \in X$ ← $f(b)$ is a max of f on X

Example: $f(x) = x^2$

$D = \mathbb{R}$

$X = [-7, 8]$



$a = 0$

$f(a) = 0$ ← min of f on $X = [-7,$

$b = 8$

$f(b) = 64$ ← max of f on X

*Note there could be more than one max or min but there exists at least one.

HW 5 #10

Let $S \subseteq \mathbb{R}$. Suppose S is compact (closed and bounded)

Prove that there exists $a, b \in S$ with $a = \inf(S)$ & $b = \sup(S)$.

Ex: $S = [-7, 8]$ is compact

$a = -7 = \inf(S)$ and $-7 \in S$

$b = 8 = \sup(S)$ and $8 \in S$

proof of HW 5 #10

Since S is bounded we know that S is bounded from below and above. So by the completeness axiom $a = \inf(S)$ and $b = \sup(S)$ exist. We need to show that $a, b \in S$.

Let's show that $b \in S$

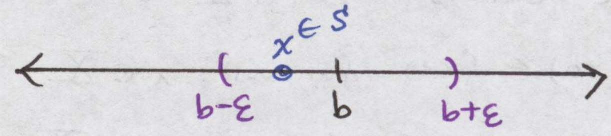
Suppose $b \notin S$

For any $\varepsilon > 0$, by the useful sup/inf fact, $\exists\, x \in S$ with

$b - \varepsilon < x \le b$

since $x \in S$ and $b \notin S$ we have

$b - \varepsilon < x < b$



So for every $\varepsilon > 0$

there exists $x \in S$ with

$x \neq b$ and $x \in (b-\varepsilon, b+\varepsilon)$. Thus, b is a limit point of S.

Since S is closed, S contains all its limit points

so $b \in S$. contradiction

So our assumption that $b \notin S$ is false.

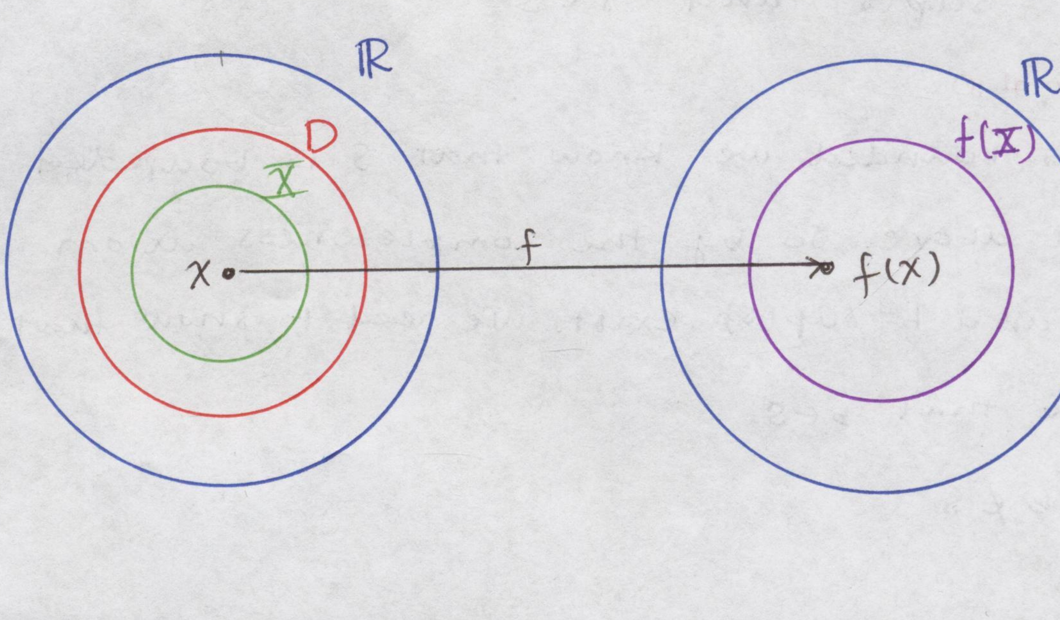
So $b \in S$.

Similarly, $a \in S$. $\boxtimes$

proof of corollary:

From a theorem from last time, since $\mathbb{X}$ is compact

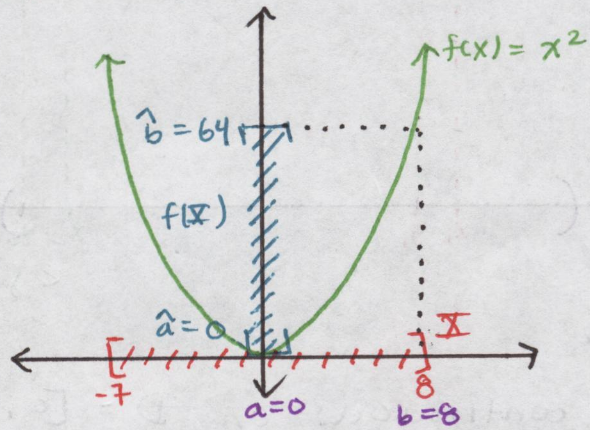
$f(\mathbb{X}) = \{ f(x) \mid x \in \mathbb{X} \}$ is compact



So, $f(\mathbb{X})$ is closed & bounded

By HW 5 #10, there exists $\hat{a}, \hat{b} \in f(\mathbb{X})$

with $\hat{a} = \inf(f(\mathbb{X}))$ and $\hat{b} = \sup(f(\mathbb{X}))$.



since $\hat{a}, \hat{b} \in f(\mathbb{X})$, there exists $a, b \in \mathbb{X}$ with

$\hat{a} = f(a)$ and $\hat{b} = f(b)$.

Since $\hat{a} = \inf(f(\mathbb{X}))$, $\hat{a} \le f(x)$ for all $x \in \mathbb{X}$

since $\hat{b} = \sup(f(\mathbb{X}))$, $\hat{b} \ge f(x)$ for all $x \in \mathbb{X}$

so $f(a) \le f(x)\ \forall\, x \in \mathbb{X}$ and $f(x) \le f(b)\ \forall\, x \in \mathbb{X}$ ☒

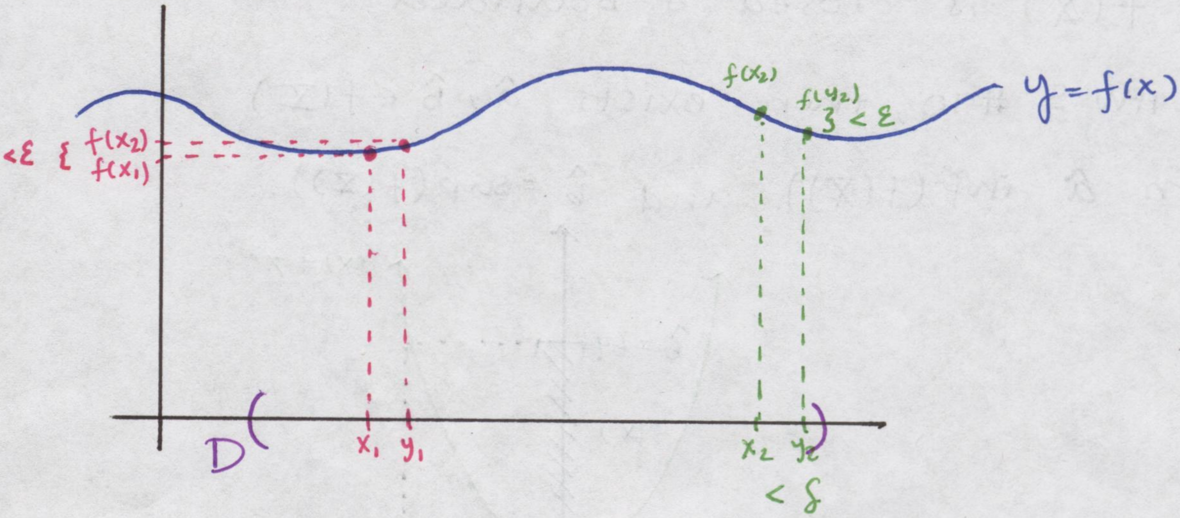
Uniform Continuity

Def: Let $f: D \to \mathbb{R}$ where $D \subseteq \mathbb{R}$.

We say that f is *uniformly continuous* on D if for every $\varepsilon > 0$ $\exists$ $\delta > 0$ so that if $x, y \in D$ and $|x - y| < \delta$ then $|f(x) - f(y)| < \varepsilon$.

Note: The same δ works over the whole set D.

let $\varepsilon = $ I ← That distance



Example: Let $f(x) = x^2$

Then f is uniformly continuous on $D = [0, 8]$

proof: Let $\varepsilon > 0$. Let $x, y \in [0, 8]$

Then, $|f(x) - f(y)| = |x^2 - y^2| = |x+y|\,|x-y|$

$$\leq (|x| + |y|)\,|x-y| \leq (8+8)\,|x-y| = 16\,|x-y|$$

Let $\delta = \frac{\varepsilon}{16}$.

If $x, y \in [0, 8]$ and $|x-y| < \delta$, then $|x^2 - y^2| \leq 16\,|x-y|$

$$< 16 \cdot \frac{\varepsilon}{16} = \varepsilon \quad \boxtimes$$

Note: $f(x) = x^2$ is continuous on all of $\mathbb{R}$

Ex: $f(x) = x^2$ is not uniformly continuous on $D = [0, \infty)$

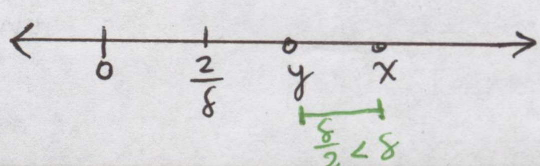
proof: Let $\varepsilon = 1$.

We want to show that for $\delta > 0$, $\exists$ $x, y \in (0, \infty)$ with $|x - y| < \delta$ but $|x^2 - y^2| \geq \varepsilon = 1$.

Suppose $\delta > 0$

Pick $x, y \in [0, \infty)$ such that $|x-y| = \frac{\delta}{2} < \delta$

and $x, y > \frac{2}{\delta} > 0$



Then $|x^2-y^2| = |x+y|\,|x-y|$

$$= (x+y)\frac{\delta}{2} > \left(\frac{2}{\delta}+\frac{2}{\delta}\right)\frac{\delta}{2} = 2 \geq 1 = \varepsilon$$

$x, y > 0 \rightarrow |x+y| = x+y$
$|x-y| = \frac{\delta}{2}$

☑

Notes: $f(x)=\sqrt{x}$
is unif. cont. on $[0,\infty)$

Theorem: Let $f: D \to \mathbb{R}$ where $D \subseteq \mathbb{R}$.

If f is uniformly continuous on D, then f is continuous on D.

Converse is NOT true

Ex: $f(x) = x^2$ is continuous on $D = [0,\infty)$
but not uniformly continuous on $D = [0,\infty)$

Proof: Let $a \in D$, let $\varepsilon > 0$.

let's show f is continuous at a

Since f is continuous on D there exists $\delta > 0$ where if $x, y \in D$ and $|x-y| < \delta$ then $|f(x)-f(y)| < \varepsilon$. Since $a \in D$, for the same δ as above, if $x \in D$ then,

$$|x-a| < \delta, \text{ then } |f(x)-f(a)| < \varepsilon$$

so f is continuous at a. ☑