Theorem: Let $D \subseteq \mathbb{R}$. Suppose that D is closed and bounded and $f: D \to \mathbb{R}$ is continuous on D.

Then f is uniformly continuous on D.

$f(x) = x^2$
cont. on $[0, \infty)$
but not unif.
cont. in $[0, \infty)$

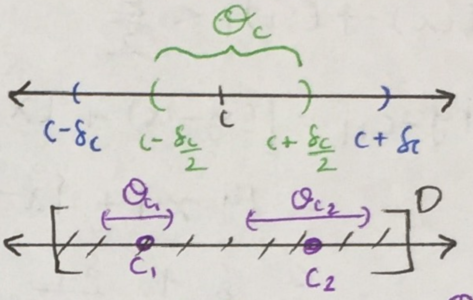
proof: Let $\varepsilon > 0$

Given $c \in D$, we know that f is continuous at c

so $\exists\ \delta_c > 0$ where if

$x \in D$ and $|x - c| < \delta_c$ then

$|f(x) - f(c)| < \frac{\varepsilon}{2}$



each $c \in D$ gets an $\mathcal{O}_c$

For each $c \in D$, define $\mathcal{O}_c = (c - \frac{\delta_c}{2}, c + \frac{\delta_c}{2})$

Let $X = \{\mathcal{O}_c \mid c \in D\}$

Then X is an open cover of D because

(1) every $c \in D$ is contained in $\mathcal{O}_c$

so, $D \subseteq \bigcup_{c \in D} \mathcal{O}_c$

(2) each $\mathcal{O}_c$ is open.

Since D is closed and bounded (i.e. compact)

$\exists$ a finite subcover $X' = \{\mathcal{O}_{c_1}, \mathcal{O}_{c_2}, \ldots, \mathcal{O}_{c_n}\}$ of D.

Suppose $x \in \mathcal{O}_{c_k}$. Then $|x - c_k| < \frac{\delta_{c_k}}{2} < \delta_{c_k}$

so, $|f(x) - f(c_k)| < \frac{\varepsilon}{2}$

Let $\delta = \min\{\frac{\delta_1}{2}, \frac{\delta_2}{2}, \frac{\delta_3}{2}, \ldots, \frac{\delta_n}{2}\}$

Suppose $x, y \in D$ and $|x-y| < \delta$. we will show $|f(x)-f(y)| < \varepsilon$ and f is uniformly continuous on D.

Since $X' = \{O_{c_1}, \ldots, O_{c_n}\}$ is an open cover of D,

we have that $x \in O_{c_i}$ for some c_i

so, $|f(x)-f(c_i)| < \frac{\varepsilon}{2}$

Also, $|y-c_i| = |(y-x) + (x-c_i)|$

$$\le |y-x| + |x-c_i|$$

$$< \delta + \frac{\delta_{c_i}}{2} \underset{\substack{\uparrow \\ \delta = \min\{\frac{\delta_{c_1}}{2}, \ldots, \frac{\delta_{c_n}}{2}\}}}{<} \frac{\delta_{c_i}}{2} + \frac{\delta_{c_i}}{2} = \delta_{c_i}$$

so, $|f(y) - f(c_i)| < \frac{\varepsilon}{2}$

Then $|f(x)-f(y)| \le |f(x)-c_i| + |c_i - f(y)|$

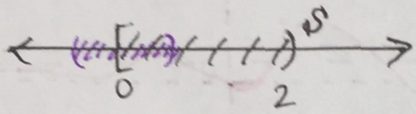
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad \boxtimes$$

P.2 12/3

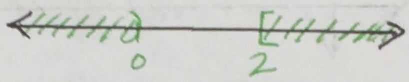
Final Exam Review

Ex: $S = [0,2)$ open? closed? compact?

open? No. $0 \in S$ but 0 is NOT an interior point of S



closed?

$\mathbb{R} \setminus S = (-\infty, 0) \cup [2, \infty)$ 

2 is in $\mathbb{R} \setminus S$ but isn't an interior pt. of $\mathbb{R} \setminus S$

so $\mathbb{R} \setminus S$ is not open. so S is not closed

compact?

S is bounded, but S is not closed so S is not compact

Ex: $S = [0,2] \cup \{4\}$

- Not open, $0, 2, 4 \in S$ but aren't interior pts. of S
- closed

$\mathbb{R} \setminus S = \underbrace{(-\infty, 0)}_{\text{open}} \cup \underbrace{(2,4)}_{\text{open}} \cup \underbrace{(4, \infty)}_{\text{open}}$, so $\mathbb{R} \setminus S$ is open

$\therefore$ S is closed

By HW $\text{open set} \cup \text{open set} = \text{open set}$ ← The union of open sets is open

- compact?

since S is closed & bounded $(S \subseteq [0,4])$

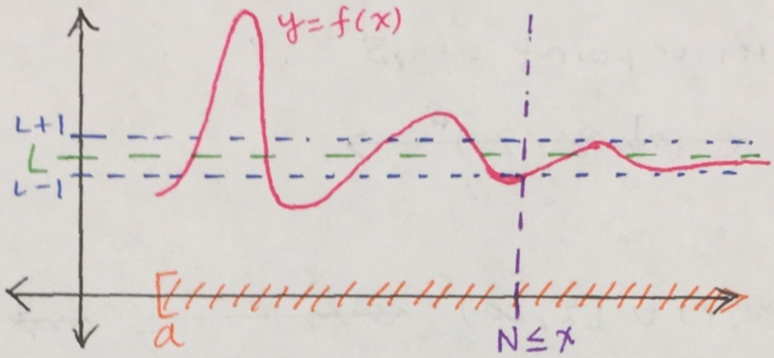
S is compact.

HW 5 #6
Don't worry about it!
:)
- No unif. cont. on Final

HW 6 #4

Let $f: [a, \infty) \to \mathbb{R}$ be continuous on $[a, \infty)$

suppose that $\lim_{x \to \infty} f(x)$ exists, then f is bounded on $[a, \infty)$



proof: Let $L = \lim_{x \to \infty} f(x)$

There exists $N > 0$ where $N \leq x$ then $|f(x) - L| < 1$

so if $N \leq x$, then $|f(x)| = |f(x) - L + L|$

$$\leq |f(x) - L| + |L| < 1 + |L|$$

since f is continuous on $[a, N]$ and $[a, N]$ is compact.

by a theorem in class f is bounded on $[a, N]$

so $\exists M > 0$ where $|f(x)| \leq M \ \forall x \in [a, N]$

because $[a, N]$ is closed & bounded.

- In class, if f is continuous on a compact set D, then $\exists c \in D$ with $f(c) \leq f(x) \ \forall x \in D$ and $d \in D$ with $f(x) \leq f(d) \ \forall x \in D$

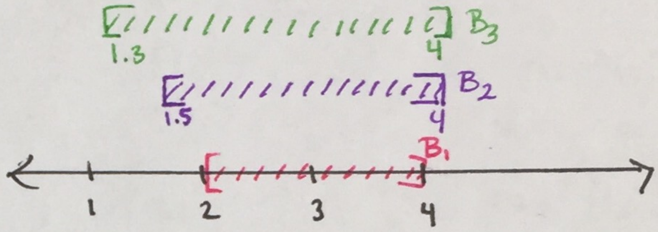
Let $\hat{M} = \max\{1 + |L|, M\}$. Then $|f(x)| \leq \hat{M} \ \forall x \in [a, \infty)$ ∎

P.3 12/3

HW 5 #9

Given an example of closed sets B_n where

$\bigcup_{n=1}^{\infty} B_n$ is not closed



Set: $B_n = [1+\frac{1}{n}, 4]$ ← each B_n is closed

$n \geq 1$

Note $\bigcup_{n=1}^{\infty} B_n = \underbrace{(1,4]}_{\text{Not closed.}}$

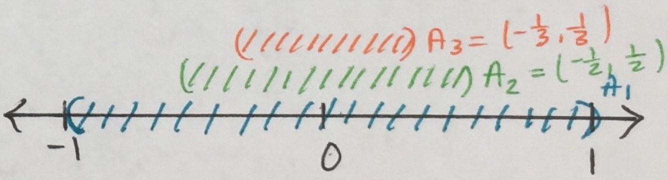
Note:

$\bigcup_{\substack{\text{any \#} \\ \text{or} \\ \text{infinite}}} (\text{open}) = \text{open}$

HW 5 #8

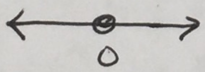
Give an example of open sets A_n where

$\bigcap_{n=1}^{\infty} A_n$ is not open.



Set: $A_n = (-\frac{1}{n}, \frac{1}{n})$, $n \geq 1$

$\bigcap_{n=1}^{\infty} A_n = \{0\}$

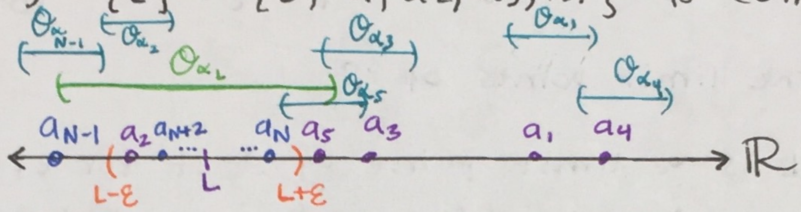


↑ Not open

HW 6 #7

Suppose that (a_n) is a sequence that converges to L.
Prove that the set

$$A = \{a_n \mid n \in \mathbb{N}\} \cup \{L\} = \{L, a_1, a_2, a_3, \ldots\} \text{ is compact.}$$



proof: Suppose that $X = \{O_\alpha\}$ is an open cover of A
so each O_α is open and $\underline{A \subseteq \bigcup O_\alpha}$ ← means: if $a \in A$ then $\exists$ at least one α with $a \in O_\alpha$

goal: We need to find a finite number of O_α that still cover A.

Since $L \in A$, $\exists\ O_{\alpha_L} \in X$ where $L \in O_{\alpha_L}$

Since O_{α_L} is open and $L \in O_{\alpha_L}$, $\exists\ \varepsilon > 0$ where $(L-\varepsilon, L+\varepsilon) \subseteq O_{\alpha_L}$

Since $\lim_{n\to\infty} a_n = L$ there exists $N > 0$ where $n \geq N$ then $\underbrace{|a_n - L| < \varepsilon}$

means: $a_n \in (L-\varepsilon, L+\varepsilon)$ for all $n \geq N$

Now for each $1 \leq i \leq N-1$, pick some $O_{\alpha_i} \in X$ where $a_i \in O_{\alpha_i}$. We can do this because X covers A.

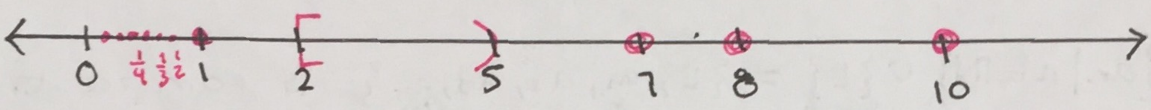
Then set $X' = \{O_{\alpha_1}, O_{\alpha_2}, \ldots, O_{\alpha_{N-1}}, O_{\alpha_L}\}$

covers: a_1, a_2, ..., a_{N-1}, L and $a_n, n \geq N$

So, X' is a finite subcover for A ☒

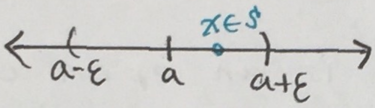
Examples of Limit points:

Ex: $S = [2,5) \cup \{7, 8, 10\} \cup \{\frac{1}{n} \mid n \geq 1\}$

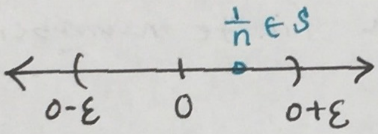


Q1: Find all the limit points of S

Def: a is a limit point of S if for every $\varepsilon > 0$
$\exists\, x \in S$ with $x \neq a$ and $a - \varepsilon < x < a + \varepsilon$

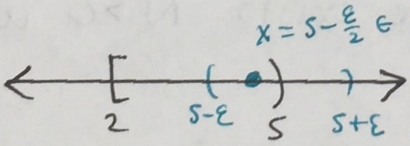


Ans: <u>0</u> is a limit point of S



given $\varepsilon > 0$, pick n where $\frac{1}{n} < \varepsilon$
i.e. $n > \frac{1}{\varepsilon}$

<u>[2,5]</u> consists of limit points of S.

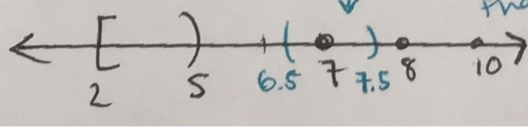


Limit points $= \{0\} \cup [2,5]$

why is 7 not a limit point?

Let $\varepsilon = \frac{1}{2}$. There are no points from S in $(7 - \frac{1}{2}, 7 + \frac{1}{2})$ that aren't equal to 7.

there is no $x \in S$ in $(6.5, 7.5)$ thats not 7.



Theorem: $S \subseteq \mathbb{R}$, $S \neq \emptyset$

a is A limit point of S iff $\exists\ (a_n)$

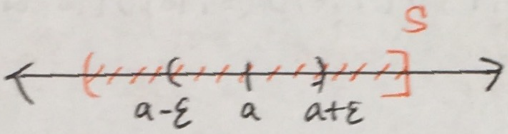
$\lim_{n\to\infty} a_n = a$ with $a_n \in S$ and $a_n \neq a\ \forall n$.

Q2: What are the interior points of S?

$S = [2,5) \cup \{7,8,10\} \cup \{\frac{1}{n} \mid n \geq 1\}$

Def: a is an interior point of S if $\exists\ \varepsilon > 0$

where $(a-\varepsilon, a+\varepsilon) \subseteq S$.



Ans: $(2,5)$

Q3: Is S open?

Ans: No $2 \in S$ and 2 is not an interior point of S

Q4: Is S closed?

Ans: No because 0 is a limit point of S

and $0 \notin S$

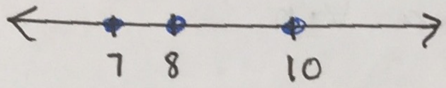
[A closed set contains all its limit points]

Q5: Is S compact?

Ans: No S is not compact since S is not closed.

Find complements

Ex: $S' = \{7, 8, 10\}$



S' is closed: (why?)

method 1: S' has no limit points. so it contains all its limit points, so S is closed.

method 2: $\mathbb{R} \setminus S' = \underbrace{(-\infty, 7)}_{\text{open set}} \cup \underbrace{(7, 8)}_{\text{open set}} \cup \underbrace{(8, 10)}_{\text{open set}} \cup \underbrace{(10, \infty)}_{\text{open set}}$

By HW 5 #3(b) If A and B are open, then $A \cup B$ is open

so $\mathbb{R} \setminus S'$ is open $\therefore S'$ is closed.

HW 6 #5(b)

If A and B are both compact, then $A \cup B$ is compact.

Proof: We first show that $A \cup B$ is closed.

We are given that A and B are closed ← since A and B are compact

consider $\mathbb{R} \setminus (A \cup B) = (\mathbb{R} \setminus A) \cap (\mathbb{R} \setminus B)$

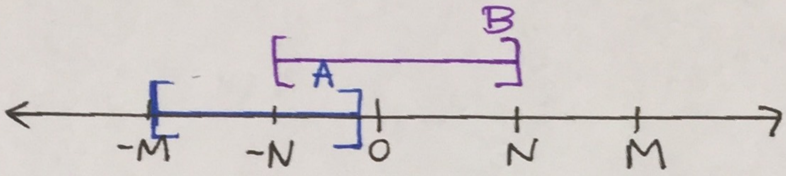
since A and B are closed, $\mathbb{R} \setminus A$ & $\mathbb{R} \setminus B$ are open.

By HW 5, 3(a), $(\mathbb{R} \setminus A) \cap (\mathbb{R} \setminus B)$ is open

so $\mathbb{R} \setminus (A \cup B)$ is open. so $A \cup B$ is closed.

we now show that $A \cup B$ is Bounded

since A and B are compact, A is bounded & B is bounded



So $\exists$ $M>0$ and $N>0$ where $-M<a<M$ and

$-N<b<N$ $\forall$ $a\in A$ and $b\in B$

let $T=\max\{M,N\}$

Then, $-T<x<T$ $\forall$ $x\in A\cup B$

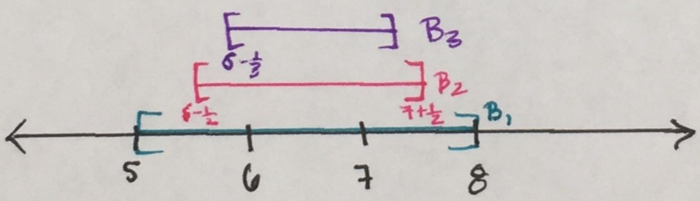
So, $A\cup B$ is bounded

Since $A\cup B$ is closed & bounded, $A\cup B$ is compact. $\boxtimes$

HW 6 #5 (d)

Given an infinite # of compact sets B_n where $\bigcap_{n=1}^{\infty} B_n$ is not compact.

set: $B_n=[6-\frac{1}{n}, 7+\frac{1}{n}]$ for $n\geq 1$ ← each of these is closed & bounded ie compact.



Ans: No such set exists

look at solution online ☺ $\boxtimes$