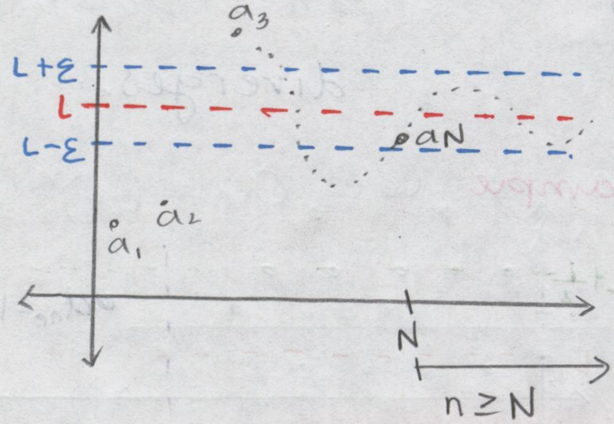
Last Time: $\lim_{n\to\infty} a_n = L$

if for every $\varepsilon > 0$ $\exists N > 0$ such that if $n \geq N$ then $|a_n - L| < \varepsilon$



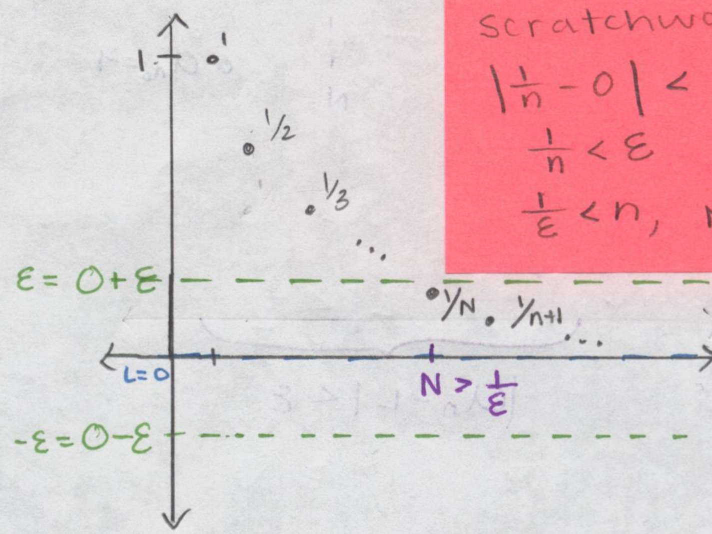
Example: $\lim_{n\to\infty} \frac{1}{n} = 0$

proof: Let $\varepsilon > 0$

pick N where $N > \frac{1}{\varepsilon}$

Then if $n \geq N$ then

$$\left|\frac{1}{n} - 0\right| \underset{\substack{\uparrow \\ n>0}}{=} \frac{1}{n} \underset{\substack{\uparrow \\ n \geq N}}{\leq} \frac{1}{N} \underset{\substack{\uparrow \\ N > \frac{1}{\varepsilon}}}{<} \varepsilon \quad \boxtimes$$



scratchwork

$\left|\frac{1}{n} - 0\right| < \varepsilon$

$\frac{1}{n} < \varepsilon$

$\frac{1}{\varepsilon} < n, \quad N > \frac{1}{\varepsilon}$

Example: Let's show that $\lim_{n\to\infty} \frac{n}{n+1} = 1$

proof: Let $\varepsilon > 0$

For any $n > 0$, note that

$$(*) \quad \left|\frac{n}{n+1} - 1\right| = \left|\frac{n}{n+1} - \frac{n+1}{n+1}\right| = \left|\frac{-1}{n+1}\right| \underset{\substack{\uparrow \\ n>0}}{=} \frac{1}{n+1}$$

scratchwork

$\frac{1}{n+1} < \varepsilon$

$\frac{1}{\varepsilon} < n+1$

$\frac{1}{\varepsilon} - 1 < n$

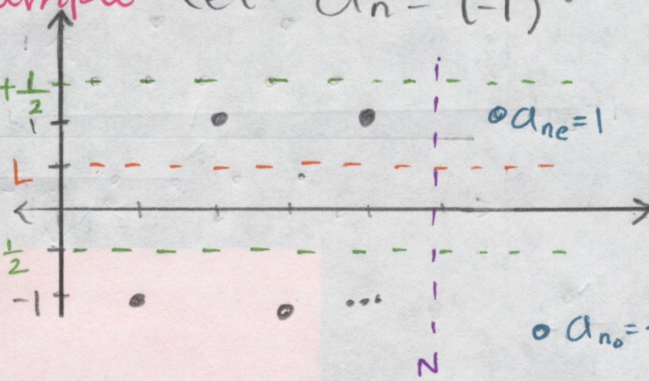
Note that $\frac{1}{n+1} < \varepsilon$ iff $\frac{1}{\varepsilon} < n+1$ iff $\frac{1}{\varepsilon} - 1 < n$.

Pick N so that $N > \frac{1}{\varepsilon} - 1$. If $n \geq N > \frac{1}{\varepsilon} - 1$, then by the previous equations $(*)$ we have:

$$\left|\frac{n}{n+1} - 1\right| = \frac{1}{n+1} < \varepsilon \quad \boxtimes$$

Def: If a_n has no limit, then we say that (a_n) **diverges.**

Example let $a_n = (-1)^n$



Let's show this sequence diverges.

Suppose $\lim_{n\to\infty} (-1)^n = L$ for some $L \in \mathbb{R}$

Pick $\varepsilon = 1/2$. Then by assumption $\exists\ N>0$ where $n \geq N$.

we have $\underbrace{|(-1)^n - L| < \frac{1}{2}}_{|a_n - L| < \varepsilon}$

Pick an even $n_e \geq N$ and an odd $n_o \geq N$.

Then $*\begin{cases} |1-L| = |(-1)^{n_e} - L| < \frac{1}{2} \\ |-1-L| = |(-1)^{n_o} - L| < \frac{1}{2} \end{cases}$ and

$|x+y| \leq |x|+|y|$

Δ-ineq.

Then $2 = |1-(-1)| = |1-L+L-(-1)| \leq |1-L| + |L-(-1)|$

$= |1-L| + |-1-L| < \frac{1}{2} + \frac{1}{2} = 1$

$|x| = |-x|$

so, $2 < 1$. contradiction ☒

Way 2: use theorem: $|x| < c$ iff $-c < x < c$

$*$ from equations above

if $|1-L| < \frac{1}{2}$ and $|-1-L| < \frac{1}{2}$

$-\frac{1}{2} < 1-L < \frac{1}{2}$ $(*)$

and $-\frac{1}{2} < -1-L < \frac{1}{2}$ ← mult (-1)

$-\frac{1}{2} < 1+L < \frac{1}{2}$ $(*)$

add $(*)$ and $(**)$

$-1 < 2 < 1$

contradiction ☒

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Theorem: Suppose (a_n) converges

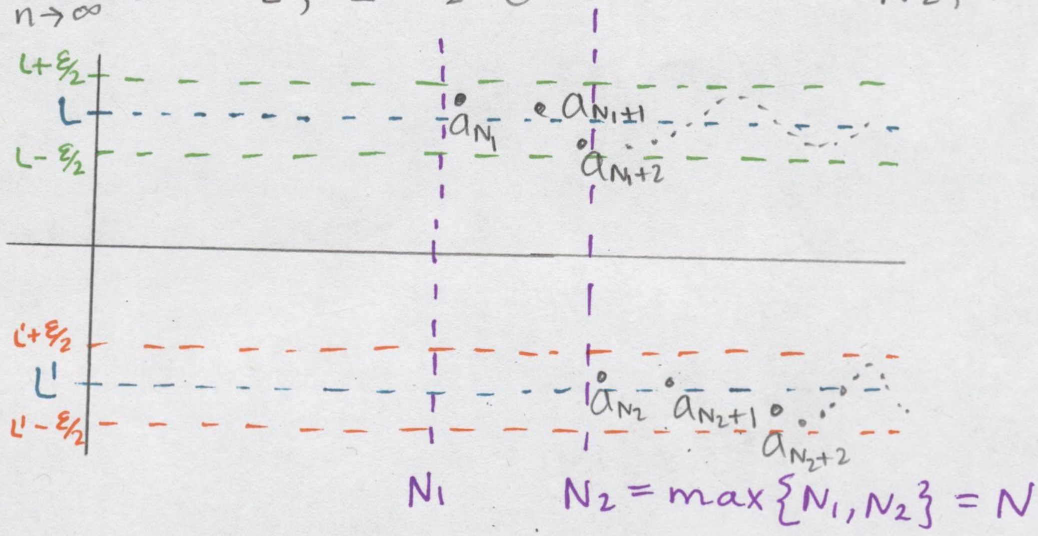
If L and L' are both limits of (a_n) then $L = L'$

proof: Let $\varepsilon > 0$

Idea: show $|L-L'| < \varepsilon$ $\forall \varepsilon > 0$, then by HW, $|L-L'| = 0$ so $L = L'$

since $\lim\limits_{n\to\infty} a_n = L$, $\exists N_1 > 0$ where if $n \geq N_1$ then $|a_n - L| < \frac{\varepsilon}{2}$

since $\lim\limits_{n\to\infty} a_n = L'$, $\exists N_2 > 0$ where if $n \geq N_2$, then $|a_n - L'| < \frac{\varepsilon}{2}$



so $n_0 > N_1$ and $n_0 \geq N_2$

thus, $|a_{n_0} - L| < \frac{\varepsilon}{2}$ and $|a_{n_0} - L'| < \frac{\varepsilon}{2}$

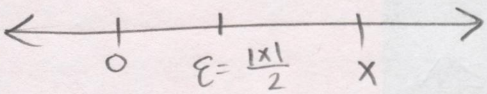
then,
$$
\begin{aligned}
|L - L'| &= |L - a_{n_0} + a_{n_0} - L'| \\
&\leq |L - a_{n_0}| + |a_{n_0} - L'| \\
&= |a_{n_0} - L| + |a_{n_0} - L'| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon
\end{aligned}
$$

So, $|L - L'| < \varepsilon$ $\forall \varepsilon > 0$. Thus, $|L - L'| = 0$ so $L = L'$ $\boxtimes$

HW#1

#2 suppose $|x| < \varepsilon \ \forall \varepsilon > 0$

If $x \neq 0$, then set $\varepsilon = \frac{|x|}{2} > 0$ and then



by assumption

$|x| < \frac{|x|}{2}$ so, $\frac{|x|}{2} < 0$

$\therefore$ contradiction