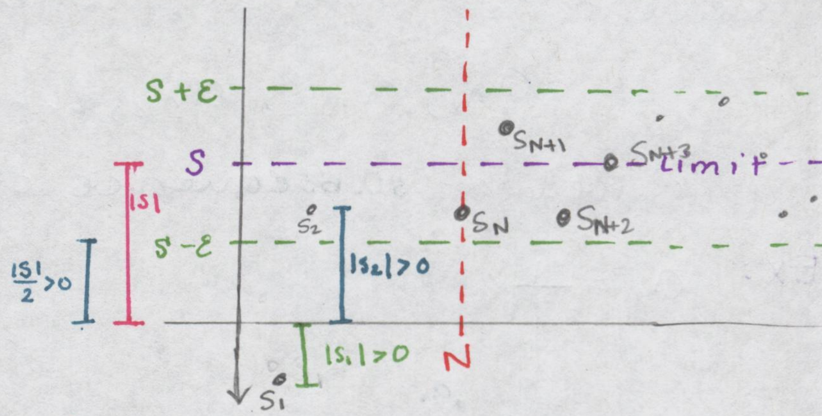
HW 2 #8

(a) Let (s_n) converge to $s \neq 0$. Assume $s_n \neq 0 \; \forall n$. Show that $\exists M > 0$ where $|s_n| \geq M \; \forall n$.



proof: Let $\varepsilon = \frac{|s|}{2} > 0$ ← since $s \neq 0$

Since $\lim_{n\to\infty} s_n = s$, $\exists N > 0$ where if $n \geq N$ then

$$|s_n - s| < \varepsilon$$

Note that if $n \geq N$ then,

$$|s| = |s - s_n + s_n| \leq |s - s_n| + |s_n| < \underbrace{\frac{|s|}{2}}_{\varepsilon} + |s_n|$$

So, $|s| < \frac{|s|}{2} + |s_n|$

Thus, $\frac{|s|}{2} < |s_n| \; \forall n \geq N$.

Let $M = \min\{|s_1|, |s_2|, \dots, |s_{N-1}|, \frac{|s|}{2}\}$

Note that $M > 0$ since $s_n \neq 0$, $s \neq 0$

If $1 \leq n \leq N-1$, then $|s_n| \geq |s_n|$

If $N \leq n$, then $|s_n| \geq \frac{|s|}{2}$

So $\forall n$, $|s_n| \geq M$ ☒

(b) If $s_n \neq 0 \; \forall n$, $s \neq 0$, and $\lim_{n\to\infty} s_n = s$, the $\lim_{n\to\infty} \frac{1}{s_n} = \frac{1}{s}$

proof: Let $\varepsilon > 0$, By part (a) $\exists M > 0$ where $|s_n| > M \; \forall n$.

Note that $\left|\frac{1}{s_n} - \frac{1}{s}\right| = \left|\frac{s - s_n}{s \cdot s_n}\right| = \frac{|s - s_n|}{|s||s_n|} < \frac{|s - s_n|}{|s| \cdot M} \; \forall n$

since $\lim_{n\to\infty} s_n = s$ $\exists N > 0$ where if $n \geq N$ then $|s - s_n| < \underbrace{\varepsilon \cdot |s| \cdot M}_{>0}$

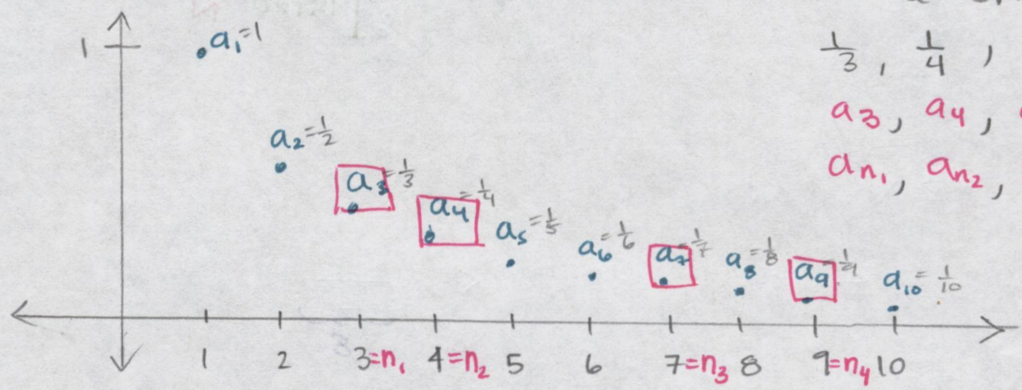
So if $n \geq N$, then $\left|\frac{1}{s_n} - \frac{1}{s}\right| < \frac{|s - s_n|}{|s| \cdot M} < \frac{\varepsilon |s| \cdot M}{|s| \cdot M} = \varepsilon$ ☒

Def: let (a_n) be a sequence of real numbers. let $n_1 < n_2 < n_3 < \dots$ be a strictly increasing sequence of natural numbers. Then the sequence,

$$a_{n_1}, a_{n_2}, a_{n_3}, \dots$$

is called a subsequence of (a_n) and denoted by (a_{n_k})

Ex: $a_n = \frac{1}{n}$



subsequence

$\frac{1}{3}, \frac{1}{4}, \frac{1}{7}, \frac{1}{9}, \dots$

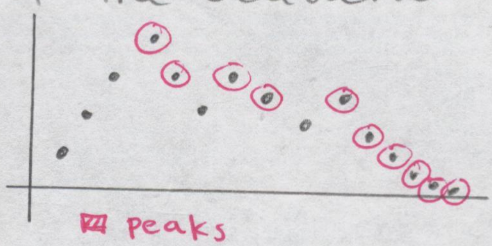
$a_3, a_4, a_7, a_9, \dots$

$a_{n_1}, a_{n_2}, a_{n_3}, a_{n_4}, \dots$

Monotone Subsequence Theorem

If (x_n) is a sequence of real numbers, then there is a subsequence of (x_n) that is monotonic.

Proof: we say that the m^{th} term x_m is a "peak" of the sequence (x_n) if $x_m \geq x_n \ \forall n \geq m$



case 1: (x_n) has infinitely many peaks.

In this case, we list our peaks:

$$x_{m_1}, x_{m_2}, x_{m_3}, \dots$$

where $m_1 < m_2 < m_3 < \dots$

Then (x_{m_k}) is monotonically decreasing.

Case 2: (x_n) has finitely many peaks (Possibly zero)

Let $s_1 = 1$ if there are no peaks.

otherwise let the peaks be $x_{m_1}, x_{m_2}, \dots, x_{m_r}$ where $m_1 < m_2 <, \dots, < m_r$.

In this case, set $s_1 = m_r + 1$. So x_{s_1} is past any peaks in the sequence. Therefore, x_{s_1} is not a peak. so $\exists\ x_{s_2}$ with $s_1 < s_2$ and $x_{s_1} < x_{s_2}$.

since x_{s_2} is not a peak $\exists\ s_3$ with $s_2 < s_3$ and $x_{s_2} < x_{s_3}$. Continue in this fashion and get a subsequence $x_{s_1}, x_{s_2}, x_{s_3}, \dots$ that satisfies

$x_{s_1} < x_{s_2} < x_{s_3} < \cdots$

So (x_{s_k}) is monotonically increasing. $\blacksquare$

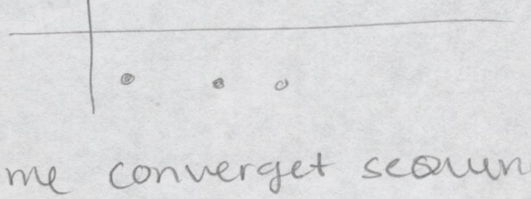
Bolzano - Weierstrass Theorem

Let (x_n) be a bounded sequence of real numbers. then $\exists$ a convergent subsequence.

proof: By the monotone subsequence thm. $\exists$ a subsequence (x_{n_k}) that is monotonic. Since (x_n) is bounded so is (x_{n_k}). Since (x_{n_k}) is bounded and monotonic, by the monotone convergence theorem,

(x_{n_k}) converges $\blacksquare$

Ex: $(x_n) = ((-1)^n)$



some converget seqeunces: 1, 1, 1, 1

-1, -1, -1, -1

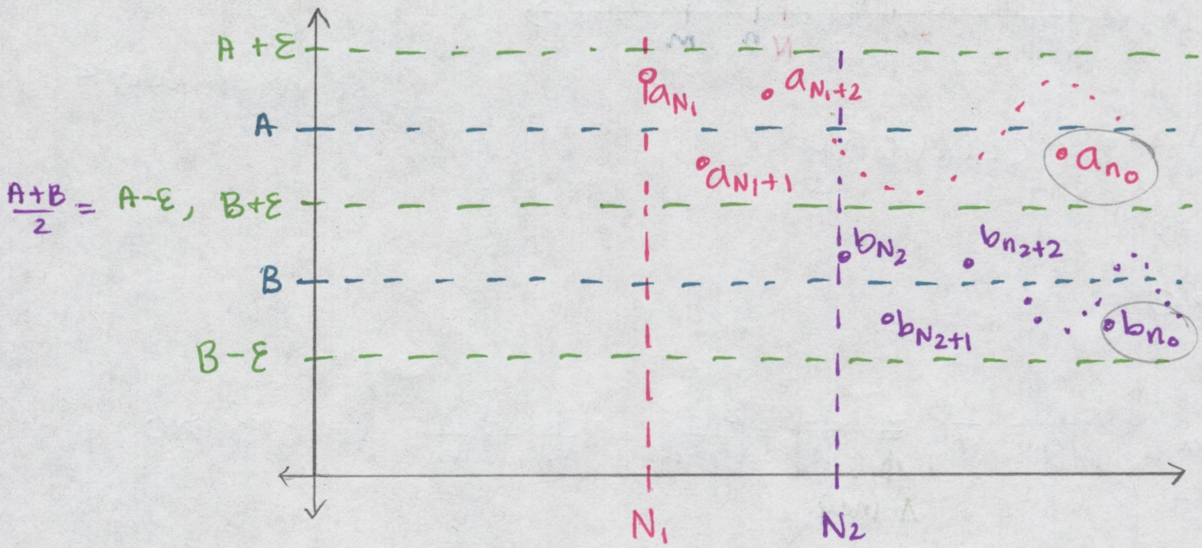
HW 2 #7 Suppose $a_n \le b_n \ \forall n$ and $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$

Then $A \le B$

proof:

Suppose $a_n \le b_n \ \forall n$ and $B < A$.

Let's show this leads to a contradiction. Let $\varepsilon = \frac{A-B}{2} > 0$



Since $\lim_{n\to\infty} a_n = A$, $\exists N_1 > 0$ where if $n \ge N_1$, then $|a_n - A| < \varepsilon$

Since $\lim_{n\to\infty} b_n = B$, $\exists N_2 > 0$ where if $n \ge N_2$, then $|b_n - B| < \varepsilon$

Let $n_0 \ge \max\{N_1, N_2\}$. Then $|a_{n_0} - A| < \varepsilon$ and $|b_{n_0} - B| < \varepsilon$

So, $-\varepsilon < a_{n_0} - A < \varepsilon$ and $-\varepsilon < b_{n_0} - b < \varepsilon$

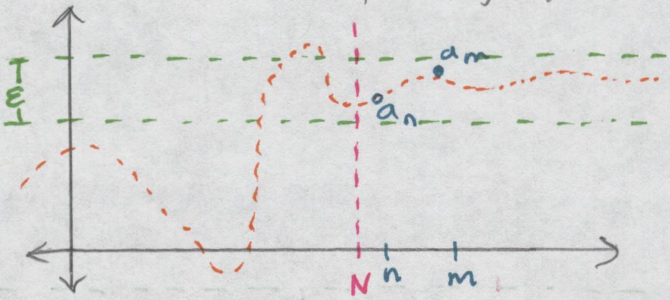
So, $\frac{B-A}{2} < a_{n_0} - A < \frac{A-B}{2}$ and $\frac{B-A}{2} < b_{n_0} - b < \frac{A-B}{2}$

$\frac{B+A}{2} < a_{n_0} < \frac{3A-B}{2}$ and $\frac{3B-A}{2} < b_{n_0} < \frac{A+B}{2}$

Then $b_{n_0} < \frac{A+B}{2} < a_{n_0}$. This contradicts that

$a_n < b_n \ \forall n \ \therefore \ A \le B$ $\boxtimes$

Def: Let (a_n) be a sequence of real numbers. We say that (a_n) is a **Cauchy sequence** if for every $\varepsilon > 0$ there exists $N > 0$ where if $n, m \geq N$ then $|a_n - a_m| < \varepsilon$



scratch work

Example: $a_n = \frac{1}{n}$ is Cauchy

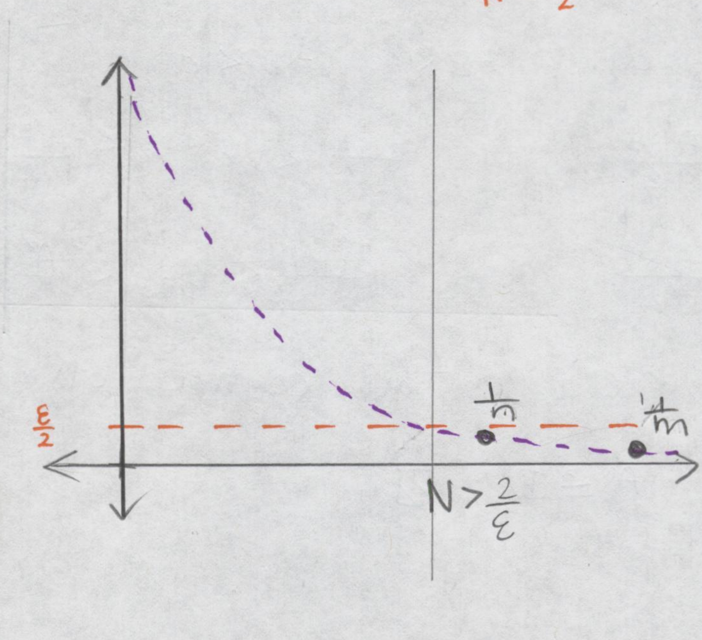
Proof: Let $\varepsilon > 0$

Then, $|a_n - a_m| = \left|\frac{1}{n} - \frac{1}{m}\right| \underset{\Delta\text{-ineq.}}{\leq} \left|\frac{1}{n}\right| + \left|\frac{-1}{m}\right| = \frac{1}{n} + \frac{1}{m}$ if $n, m > 0$

Pick $N > \frac{2}{\varepsilon}$

If $n, m \geq N$, then,

$$\left|\frac{1}{n} - \frac{1}{m}\right| \leq \frac{1}{n} + \frac{1}{m} \leq \frac{1}{N} + \frac{1}{N} \underset{\substack{N > \frac{2}{\varepsilon} \\ \frac{1}{N} < \frac{\varepsilon}{2}}}{<} \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad \square$$



Theorem: Let (a_n) be a sequence of real numbers. Then (a_n) converges to a limit in $\mathbb{R}$ iff (a_n) is a Cauchy sequence.

proof: ($\Leftarrow$) Suppose (a_n) is a Cauchy sequence of real #s

HW: If (a_n) is Cauchy, then (a_n) is bounded.

This step uses the completness axiom

$\rightarrow$ By Bolzano-Weierstrass (a_n) has a convergent subsequence. So there exists a subsequence (a_{n_k}) that converges to some $L \in \mathbb{R}$

Let's show $L = \lim_{n\to\infty} a_n$

since (a_n) is Cauchy;

$\exists N > 0$ where if $n, m > N$ then

$|a_n - a_m| < \frac{\varepsilon}{2}$

since $\lim_{k\to\infty} a_{n_k} = L$, $\exists \hat{n}_k \geq N$

where $|a_{\hat{n}_k} - L| < \frac{\varepsilon}{2}$. so if $n \geq N$ then,

$$|a_n - L| = |a_n - a_{\hat{n}_k} + a_{\hat{n}_k} - L| \leq |a_n - a_{\hat{n}_k}| + |a_{\hat{n}_k} - L| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

scratch work

$$|a_n - L| < |a_n - a_{n_k} + a_{n_k} - L|$$
$$\leq |a_n - a_{n_k}| + |a_{n_k} - L|$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

($\Rightarrow$) suppose (a_n) converges to $L \in \mathbb{R}$

let $\varepsilon > 0$ then $\exists N > 0$ where if $n \geq N$ then

$|a_n - L| < \frac{\varepsilon}{2}$. Then if $n, m \geq N$ then

$$|a_n - a_m| = |a_n - L + L - a_m|$$
$$\leq |a_n - L| + |L - a_m|$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

So (a_n) is Cauchy $\boxtimes$