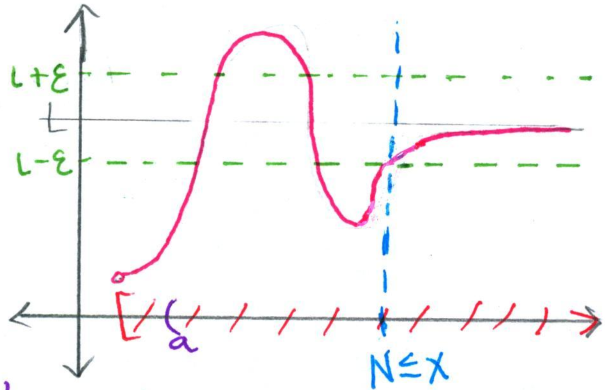
HW 3 Stuff

Def: Let $f: D \to \mathbb{R}$ where $D \subseteq \mathbb{R}$ and D contains some interval (a, ∞) for some $a \in \mathbb{R}$



Let $L \in \mathbb{R}$. We say that the limit of f as x tends to infinity is L and write $\lim_{n\to\infty} f(x) = L$ if $\forall\, \varepsilon > 0\ \exists\, N > 0$ where if $x \in D$ and $x \geq N$ then $|f(x) - L| < \varepsilon$.

Example: $f(x) = \frac{2x-3}{5x+10}$. Let's show $\lim_{x\to\infty} \frac{2x-3}{5x+10} = \frac{2}{5}$

Proof:

Let $\varepsilon > 0$, Note that

$$\left|\frac{2x-3}{5x+10} - \frac{2}{5}\right| = \left|\frac{10x-15-10x-20}{5(5x+10)}\right| \underset{\substack{\uparrow \\ x\to\infty \\ \text{assume } x>0}}{=} \frac{7}{5x+10}$$

Note that $\frac{7}{5x+10} < \varepsilon$ iff $\frac{7}{\varepsilon} < 5x+10$ iff $\left(\frac{\frac{7}{\varepsilon}-10}{5}\right) < x$

Pick $N > \left(\frac{\frac{7}{\varepsilon}-10}{5}\right)$

If $x \geq N > \left(\frac{\frac{7}{\varepsilon}-10}{5}\right)$, then $\left|\frac{2x-3}{5x+10} - \frac{2}{5}\right| < \varepsilon$

Example: Suppose $f: D \to \mathbb{R}$ and $g: D \to \mathbb{R}$ for some $D \subseteq \mathbb{R}$ that contains an interval of the form (a, ∞)

Assume $\lim_{x\to\infty} f(x) = F$ and $\lim_{x\to\infty} g(x) = G$

If $\alpha, \beta \in \mathbb{R}$ then,

$$\lim_{x\to\infty} [\alpha f(x) + \beta g(x)] = \alpha F + \beta G.$$

proof: Let $\varepsilon > 0$

It is true that,

$$|(\alpha f(x)+\beta g(x)) - (\alpha F + \beta G)| = |\alpha(f(x)-F) + \beta(g(x)-G)|$$

$$\leq |\alpha(f(x)-F)| + |\beta(g(x)-G)| = |\alpha||f(x)-F| + |\beta||g(x)-G|$$

$$< (|\alpha|+1)|f(x)-F| + (|\beta|+1)|g(x)-G|$$

in case $|\alpha|=0$ in case $|\beta|=0$

Since $\lim_{x\to\infty} f(x) = F$ $\exists N_1 > 0$ where if $x \geq N_1$ then $|f(x)-F| < \frac{\varepsilon}{2(|\alpha|+1)}$

since $\lim_{x\to\infty} g(x) = G$ $\exists N_2 > 0$ where if $x \geq N_2$ then $|g(x)-G| < \frac{\varepsilon}{2(|\beta|+1)}$

Let $N = \max\{N_1, N_2\}$. If $x \geq N$, then

$$|(\alpha f(x)+\beta g(x)) - (\alpha F - \beta G)| < (|\alpha|+1)|f(x)-F| + (|\beta|+1)|g(x)-G|$$

$$< (|\alpha|+1)\frac{\varepsilon}{2(|\alpha|+1)} + (|\beta|+1)\frac{\varepsilon}{2(|\beta|+1)}$$

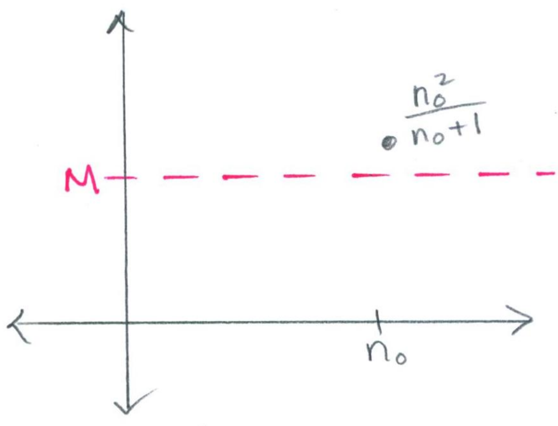
$$= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad \square$$

Homework 2 #3(h)

Show that $\lim_{n\to\infty} \frac{n^2}{n+1}$ does not exist.

proof: we can show that $\frac{n^2}{n+1}$ is unbounded and then the limit won't exist.

Let $M > 0$, we will show that $\exists n_0 > 0$ where $\frac{n_0^2}{n_0+1} > M$



Note that if $n \geq 1$ then

$$\frac{n^2}{n+1} \geq \frac{n^2}{n+n} = \frac{n^2}{2n} = \frac{n}{2}$$

and $\frac{n}{2} > M$ iff $n > 2M$

If $n_0 > 2M$ then,

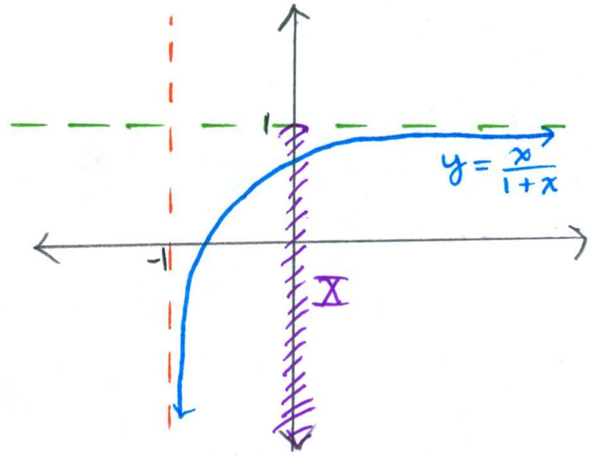
$$\frac{n_0^2}{n_0+1} \geq \frac{n_0}{2} > M \quad \square$$

Homework 1 #1(d)

Find inf(X) and sup(X), if they exist

$$X = \left\{ \frac{x}{1+x} \;\middle|\; x > -1 \right\}$$

$$X = (-\infty, 1)$$



Inf(X) does not exist

sup(X) = 1

Homework 1 #1(e)

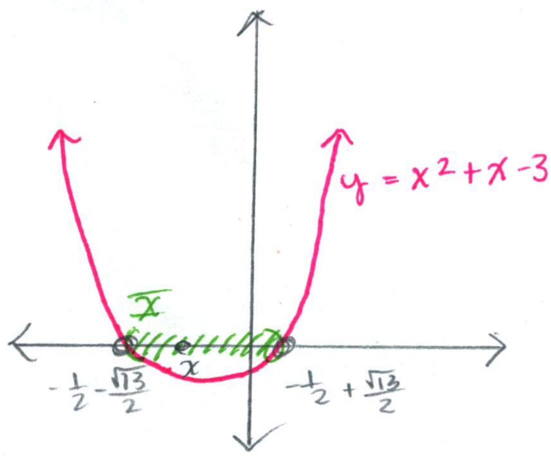
Find inf(X) and sup(X), if they exist

$$X = \{ x \mid x^2 + x < 3 \}$$

$$y = x^2 + x - 3$$

$$x^2 + x - 3 = 0$$

$$x = \frac{-1 \pm \sqrt{1^2 - 4(1)(-3)}}{2(1)} = \frac{-1 \pm \sqrt{13}}{2} = \frac{-1}{2} \pm \frac{\sqrt{13}}{2}$$



smallest $y' = (x^2 + x - 3)' = 0$

$$2x + 1 = 0$$

$x = -1/2$ ← min

$$\inf(X) = -\frac{1}{2} - \frac{\sqrt{13}}{2}$$

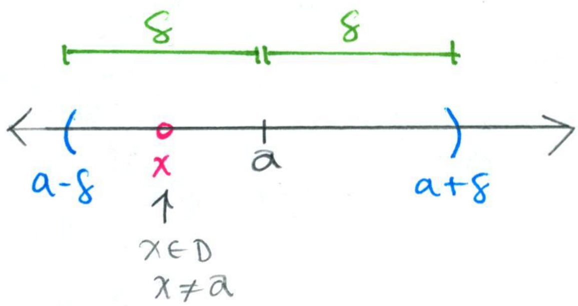
$$\sup(X) = -\frac{1}{2} + \frac{\sqrt{13}}{2}$$

Limit of a function at $x = a$

HW #3
δ - delta
ε - epsilon

Def: Let $D \subseteq \mathbb{R}$

We say that a is a **limit point** (or cluster pt., or accumulation pt) if for every $\delta > 0$ $\exists$ $x \in D$ with $0 < |x - a| < \delta$ (this means $x \neq a$ and $|x - a| < \delta$)

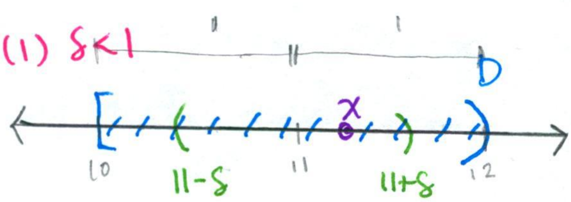


Idea: a is a limit point of D if there are points in D, not equal to a, that are getting closer and closer to a.

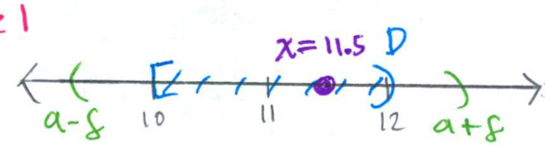
Try to prove: a is a limit point of D iff $\exists$ a sequence (x_n) where $x_n \to a$ and $x_n \in D$ $\forall n$.

Ex: $D = [10, 12)$, is 11 a limit point of D?

(1) $\delta < 1$



(2) $\delta \geq 1$



For any $\delta > 0$ we can pick $x \in D$, $x \neq a$ as follows:

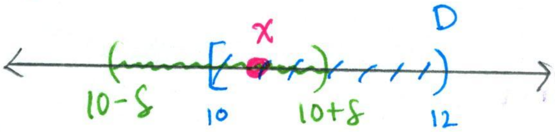
(1) If $\delta < 1$, then pick x to be the midpoint between 11 and $11 + \delta$. Then $x \in D$ and $x \neq a$ and $|x - a| = |x - 11| < \delta$.

(2) If $\delta \geq 1$, then set $x = 11.5$
then $x \in D$, $x \neq 11$ and $|x - a| < \delta$ ☒

Ans: yes, 11 is a limit point of D.

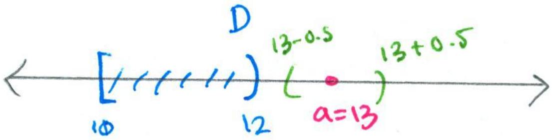
Ex: Is 10 a limit point of D? Ans. yes

$\delta < 2$



- Pick x half way between 10 and $10+\delta$ (if $\delta<2$)
- Other pick $x = 11$ if $\delta \geq 2$

13 is not a limit point



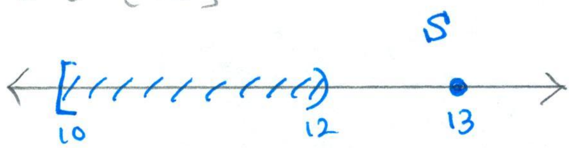
- Set $\delta = 0.5$

There is no $x \in D$ with

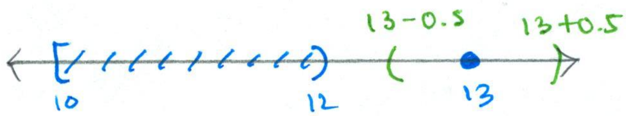
$|x-13| < 0.5$

$13 - 0.5 < x < 13 + 0.5$

Example: $S = [10,12] \cup \{13\}$



Is 13 a limit point of S?

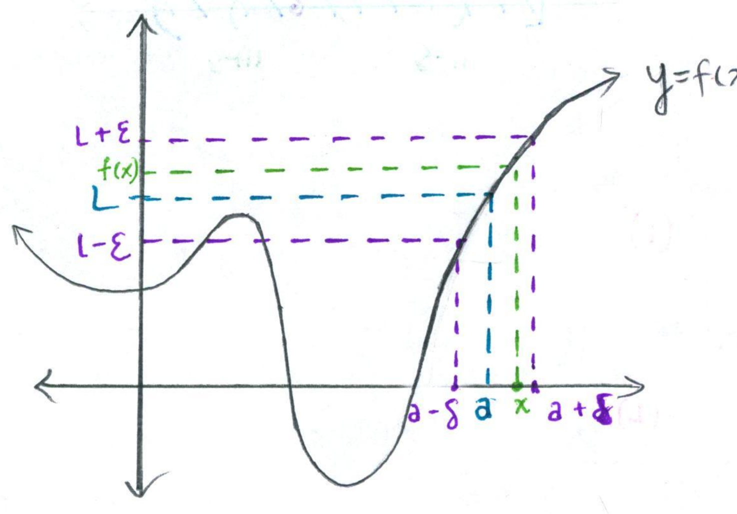


Set $\delta = 0.5$. Then no $x \in S$, $x \neq 13$, with $|x-13| < 0.5$

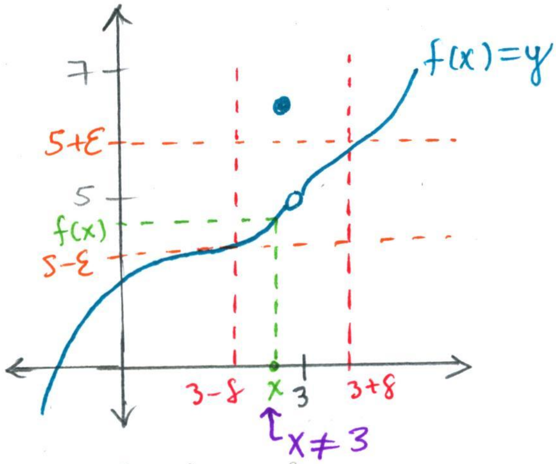
Def: Let $D \subseteq \mathbb{R}$, Let $f: D \to \mathbb{R}$.

Let a be a limit point of D

We say that f has a limit as x approaches a if $\exists\, L \in \mathbb{R}$ where for every $\varepsilon > 0$ $\exists\, \delta > 0$ such that if $x \in D$ and $\underbrace{0 < |x-a| < \delta}_{x \neq a \text{ and } |x-a|<\delta}$ then $|f(x) - L| < \varepsilon$



Notation: We write $\lim_{x \to a} f(x) = L$ if this is the case.

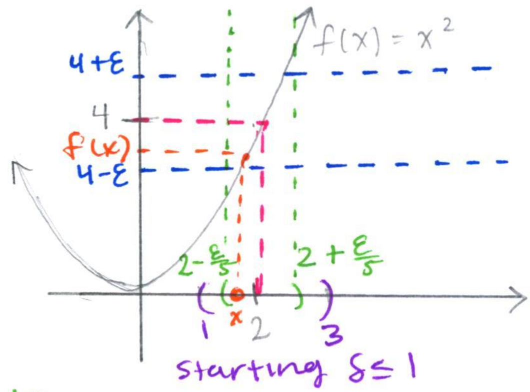


$$\lim_{x\to 3} f(x) = 5$$

Example: let $f(x) = x^2$. Let's show that $\lim_{x\to 2} x^2 = 4$

proof: let $\varepsilon > 0$

(Note that $D = \mathbb{R}$ in this case)



Goal: Find $\delta > 0$ where if

$0 < |x-2| < \delta$, then $|x^2-4| < \varepsilon$

Note that, $|x^2-4| = \underbrace{|x-2|}_{\text{we will be able to control this part via } \delta.}\ \underbrace{|x+2|}_{\text{we need to bound this part to remove it.}}$

Suppose $\delta \le 1$. Suppose $0 < |x-2| < \delta \le 1$.

Then $-1 < x-2 < 1$ ← The same as $|x-2|<1$

Add 4 to obtain $3 < x+2 < 5$

So if $0 < |x-2| < 1$, then $|x+2| < 5$ * starting bound

so if $0 < |x-2| < 1$, then $|x^2-4| = |x-2||x+2| < 5|x-2|$

set $\delta = \min\{1, \frac{\varepsilon}{5}\}$

starting bound

If $0 < |x-2| < \delta$, then

$0<|x-2|<\delta \le \frac{\varepsilon}{5}$

$|x^2-4| = |x-2||x+2| < 5|x-2| < 5 \cdot \frac{\varepsilon}{5} = \varepsilon$ ☐

$0 < |x-2| < \delta \le 1$