Recall: $f: D \to \mathbb{R}$, $D \subseteq \mathbb{R}$

is continuous at $a \in D$ if for every $\varepsilon > 0$ $\exists\, \delta > 0$ where if $x \in D$ and $|x-a| < \delta$ then $|f(x) - f(a)| < \varepsilon$.

Example: Let $f: \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$

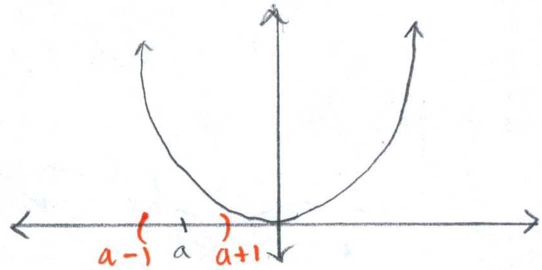
We will show that f is continuous on all of $\mathbb{R}$.

proof: Let $a \in \mathbb{R}$

We will show that $f(x) = x^2$ is continuous at a.

let $\varepsilon > 0$

Note that $|x^2 - a^2| = \underbrace{|x+a|}_{\text{bound with starting bound on } \delta} \underbrace{|x-a|}_{\text{bound this via } \delta}$



Start by assuming $\delta \le 1$. Then if $\boxed{|x-a| < \delta \le 1}$ we have $|x+a| = |x-a+a+a| = |x-a+2a|$

$\le |x-a| + |2a| < 1 + 2|a|$

so if $|x-a| < 1$, then,

$|x^2 - a^2| = |x+a||x-a| < (1+2|a|)|x-a|$

let $\delta = \min\left\{1, \frac{\varepsilon}{1+2|a|}\right\}$

If $|x-a| < \delta$, then $|x^2 - a^2| = |x+a||x-a| \underset{|x-a|<1}{<} (1+2|a|)|x-a|$

$\underset{|x-a| < \frac{\varepsilon}{1+2|a|}}{<} (1+2|a|) \dfrac{\varepsilon}{1+2|a|} = \varepsilon$ ☑

- Note, in this case we assume $\delta \le 1$, which works because $f(x) = x^2$ is defined in all $\mathbb{R}$.

However if $f(x) =$

Theorem: Let $D \subseteq \mathbb{R}$

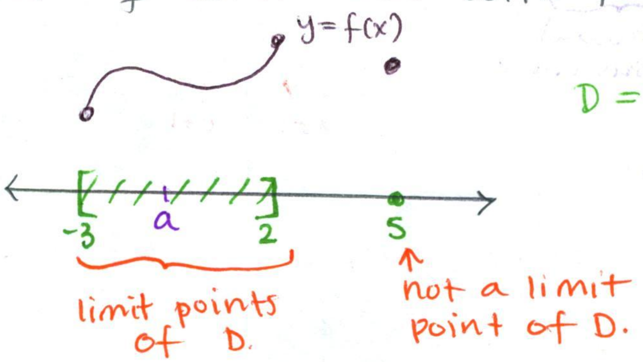
Let $a \in D$ and $\alpha \in \mathbb{R}$

Let $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$

If f and g are continuous at a then so are αf, $f+g$, $f-g$, and fg. If $f(a) \neq 0$ and f is continuous at a, then $\frac{1}{f}$ is continuous at a.

proof:

If a is not a limit point of D, then all the functions αf, $f+g$, $f-g$, fg, $\frac{1}{f}$ are continuous at a (with $\frac{1}{f}$ continuous with $f(a) \neq 0$)



$D = [-3, 2] \cup \{5\}$

f is continuous at 5

Suppose a is a limit point of D. Suppose f and g are continuous at a. Let's show fg is continuous at a

since a is a limit point of D and f and g are continuous at a, we get

$$\lim_{x \to a} f(x) = f(a) \quad \text{and} \quad \lim_{x \to a} g(x) = g(a)$$

By thms on limits,

$$\lim_{x \to a} (f(x)g(x)) \underset{\uparrow}{=} \left(\lim_{x \to a} f(x)\right)\left(\lim_{x \to a} g(x)\right) = f(a) \cdot g(a)$$

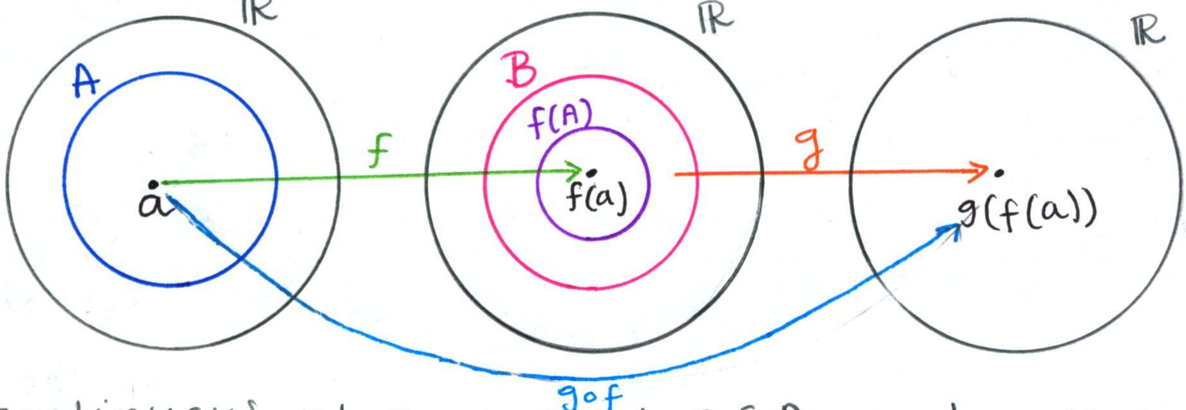
limit thm.

So, fg is continuous at a. ☒

same idea for αf, $f \pm g$, $\frac{1}{f}$.

P.2 10/15

Theorem: Let $A, B \subseteq \mathbb{R}$ and $f: A \to \mathbb{R}$ and $g: B \to \mathbb{R}$ and $\underbrace{f(A)}_{\text{image or range of } f} \subseteq B$



If f is continuous at some point $a \in A$, and g is cont. at $f(a)$, then $g \circ f: A \to \mathbb{R}$ is continuous at a.

proof: Let $\varepsilon > 0$.

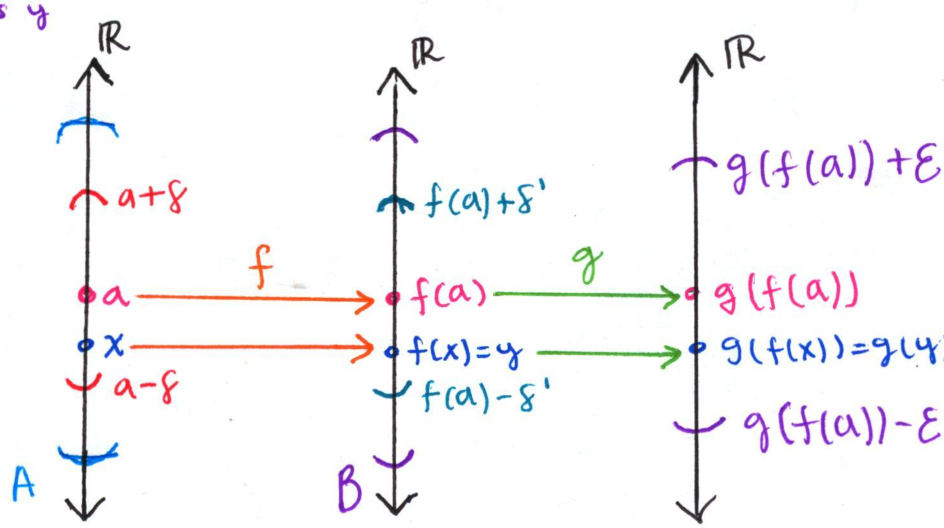
We need to find $\delta > 0$ where if $x \in A$ and $|x - a| < \delta$ then $|(g \circ f)(x) - (g \circ f)(a)| < \varepsilon$

Since g is continuous at $f(a)$, $\exists\ \delta' > 0$ where if $y \in B$ and $|y - f(a)| < \delta'$ then $|g(y) - g(f(a))| < \varepsilon$

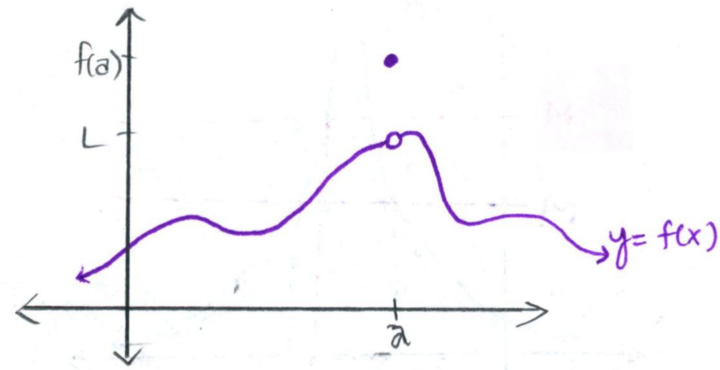
Since f is continuous at a, $\exists\ \delta > 0$ where if $x \in A$ and $|x - a| < \delta$, then $|f(x) - f(a)| < \delta'$

So if $x \in A$ and $|x - a| < \delta$ then $|\underbrace{f(x)}_{\text{think of this as } y} - f(a)| < \delta'$

where $\underbrace{f(x)}_{\text{think of this as } y} \in f(A) \subseteq B$. and so $|g(f(x)) - g(f(a))| < \varepsilon$ ☒



HW 3 #2(a)



Let $D \subseteq \mathbb{R}$, Let $f: D \to \mathbb{R}$

let a be a limit point of D.

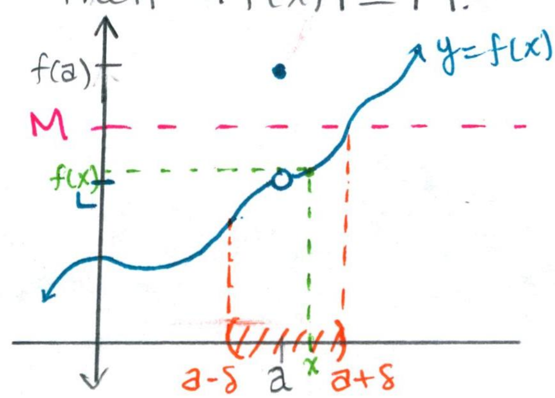
Suppose $\lim_{x \to a} f(x)$ exists.

Then f is "bounded near a"

That is, there exists $M > 0$ and $\delta > 0$ where if $x \in D$ and $\underbrace{0 < |x-a| < \delta}$ then $|f(x)| \leq M$.

$x \neq a$

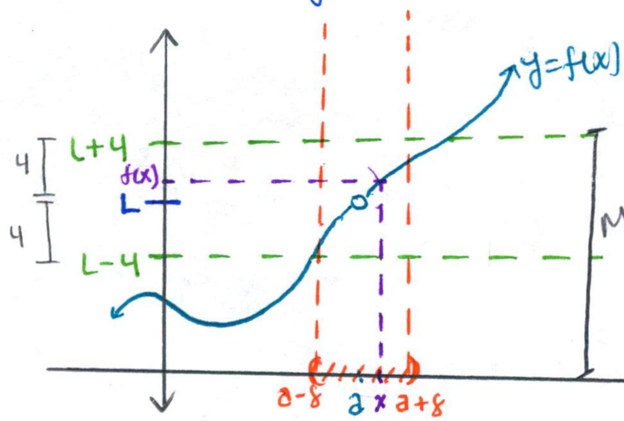
$a-\delta < x < a+\delta$



Proof: Let $\varepsilon = 4$ ← you can pick any positive # you want.

Let $\lim_{x \to a} f(x) = L$, Then there exists $\delta > 0$ where if $x \in D$ and $0 < |x-a| < \delta$, then $|f(x) - L| < 4$

so if $x \in D$ and $0 < |x-a| < \delta$, then,



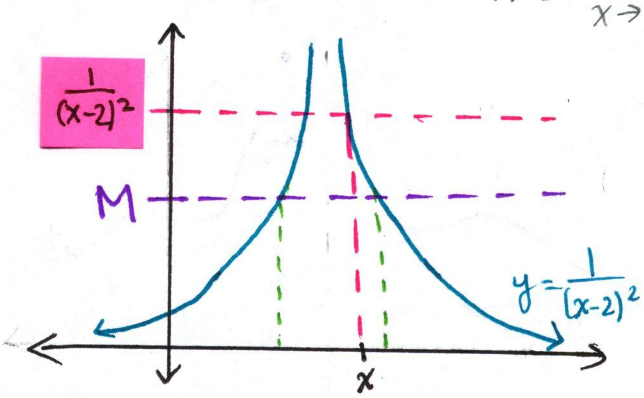
$$|f(x)| = |f(x) - L + L| \underset{\uparrow \,\Delta\text{-ineq.}}{\leq} |f(x) - L| + |L| < 4 + |L|$$

Set $M = 4 + |L|$ and we are done.

$\square$

HW 3 #2(b)

Show that the $\lim_{x\to 2} \frac{1}{(x-2)^2}$ does not exist.

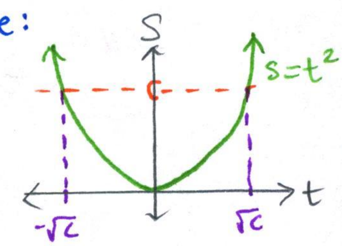


Proof: We show that (a) is not true for $f(x)=\frac{1}{(x-2)^2}$ at $a=2$.

Let $M>0$,

Note that $\left|\frac{1}{(x-2)^2}\right|>M$ iff $\frac{1}{(x-2)^2}>M$

Note:



$t^2 < c$

$-\sqrt{c} < t < \sqrt{c}$

iff $\frac{1}{M} > (x-2)^2$ and $x \neq 2$

iff $-\frac{1}{\sqrt{M}} < (x-2) < \frac{1}{\sqrt{M}}$ and $x \neq 2$

iff $0<|x-2|<\frac{1}{\sqrt{M}}$

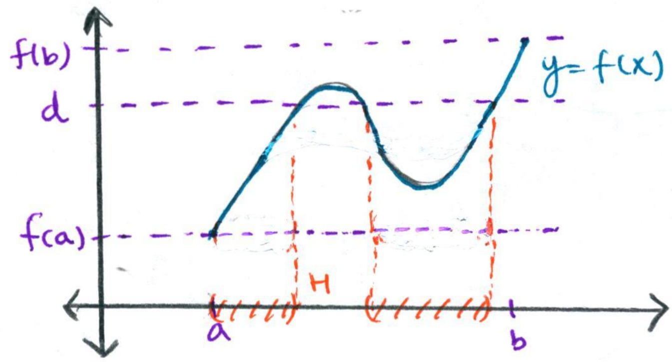
So if $0<|x-2|<\frac{1}{M}$ then $\left|\frac{1}{(x-2)^2}\right|>M$

There is no $\delta>0$ where $0<|x-2|<\delta$ implies $\left|\frac{1}{(x-2)^2}\right| \leq M$

since M is arbitrary, $\frac{1}{(x-2)^2}$ does not satisfy part (a) at $a=2$. So, $\lim_{x\to 2}\frac{1}{(x-2)^2}$ does not exist. $\boxtimes$

Intermediate Value Theorem

Let f be continuous on $D = [a, b]$ and suppose $f(a) < f(b)$. Given d with $f(a) < d < f(b)$ $\exists\, c$ with $a < c < b$ with $f(c) = d$

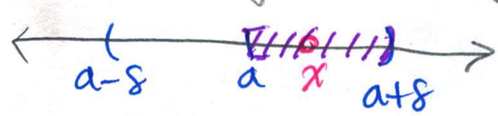


Proof: Let $H = \{x \in (a,b) \mid f(x) < d\}$

We first show that $\sup(H)$ exists.

Let $\varepsilon = d - f(a) > 0$. since f is continuous at a, $\exists\, \delta > 0$ where if $x \in [a,b]$ and $|x-a| < \delta$, then $|f(x) - f(a)| < \varepsilon = d - f(a)$

we may assume that $\delta < b-a$ by shrinking δ if necessary so that way $a + \delta < b$.



$\leftarrow$ $\boxed{\begin{array}{c} x \in [a,b] \\ |x-a| < \delta \end{array}}$

so if $x \in (a, a+\delta)$

then $f(x) - f(a) \leq |f(x) - f(a)| < d - f(a)$.

so if $x \in (a, a+\delta)$, then $f(x) < d$.

Therefore, $(a, a+\delta) \subseteq H$, Thus $H \neq \emptyset$

Also, b is an upper bound for H so, by the completeness axiom, $\sup(H)$ exists. let $c = \sup(H)$. Since b is an upper bound for H, and $c = \sup(H)$, we have $c \leq b$

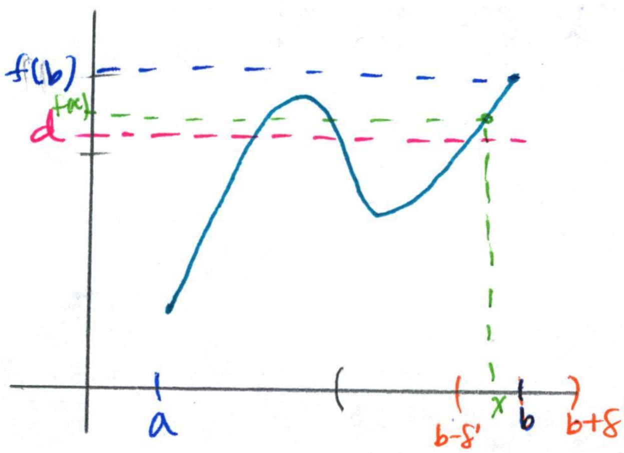
why does $c \neq b$?

since f is continuous at b $\exists\, \delta' > 0$ where if $x \in [a,b]$ and $|x-b| < \delta'$ then $|f(x)-f(b)| < f(b)-d$

so, if $x \in [a,b]$ and $|x-b|<\delta'$ then

$$\underbrace{-(f(b)-d) < f(x)-f(b)} < f(b)-d$$

so if $x \in [a,b]$ and $|x-b| < \delta'$ then $d < f(x)$



we may assume $\delta' < b-a$ so that $a < b-\delta'$

Then $b-\delta'$ is an upper bound for H. so,

$c \leq b-\delta' < b$. Thus $a < c < b$

Next time we show $f(c) = d$