California State University – Los Angeles

Department of Mathematics

Master's Degree Comprehensive Examination

Analysis Fall 2022

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Do at least five of the following seven problems. All problems count equally. If you attempt more than five, the best five will be used.

(1) Write in a fairly soft pencil (number 2) (or in ink if you wish) so that your work will duplicate well. There should be a supply available.
(2) Write on one side of the paper only.
(3) Begin each problem on a new page.
(4) Assemble the problems you hand in in numerical order.

Exams are graded anonymously, so put your name only where directed and follow any instructions concerning identification code numbers.

SECTION 1 – Do two (2) problems from this section. If you attempt all three, then the best two will be used for your grade.

Fall 2022 #1. Consider the sequence defined by $x_0 = 1$ and

$$x_{n+1} = 1 + \frac{1}{x_n}$$

for all integers $n \geq 0$.

(a) Show that the sequence satisfies

$$1 \leq x_n \leq 2$$

for all non-negative integers n.

(b) Prove that (x_n) has a convergent subsequence (x_{n_k}). Hint: Use your answer from (a).

SOLUTION:

(a) We prove this by induction. For the $n = 0$ case, we have

$$x_0 = 1,$$

so the result obviously holds for $n = 0$. Assume now that the result holds for $n = k$, so that

$$\frac{1}{2} \leq \frac{1}{x_k} \leq 1.$$

It follows that

$$\frac{3}{2} \leq 1 + \frac{1}{x_k} \leq 2,$$

from which we deduce that

$$\frac{3}{2} \leq x_{k+1} \leq 2.$$

Thus, the result holds for the $n = k + 1$ case. By the principle of mathematical induction, it follows that the result holds for all $n \geq 0$.

(b) Since the sequence is bounded, the Bolzano-Weierstrass theorem implies that there exists a convergent subsequence x_{n_k}.

Fall 2022 #2. Let $(x_n)_{n=1}^{\infty}$ be a sequence of real numbers.

(a) Define what it means for $(x_n)_{n=1}^{\infty}$ to be a "Cauchy sequence."

(b) Use your answer from (a) to prove that if $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence, then $\{x_n \mid n \in \mathbb{N}\}$ is bounded. (Here $\mathbb{N}$ denotes the set of positive integers.)

SOLUTIONS:

https://proofwiki.org/wiki/Cauchy_Sequence_is_Bounded/Real_Numbers

Fall 2022 #3.

Let $f : D \to \mathbb{R}$ be a continuous function on an open interval D. Prove that the function $f_+ : D \to \mathbb{R}$ defined by

$$f_+(x) = \max\{f(x), 0\}$$

is continuous.

SOLUTIONS:

Solution #1:

https://www.reddit.com/r/learnmath/comments/s1eshq/show_that_if_fg_are_continuous_then_maxfg_is/

Solution #2:

This is easy to see if we re-write the function f_+ as follows:

$$f_+(x) = \max\{f(x), 0\} = \frac{f(x) + |f(x)|}{2}.$$

SECTION 2 – Do three (3) problems from this section. If you attempt more than three, then the best three will be used for your grade.

Fall 2022 #4. Let $T : C([0,1]) \to \mathbb{R}$ be the bounded linear transformation defined by

$$T(f) = \int_0^1 f(x)\ dx.$$

Here $C([0,1])$ denotes the space of continuous functions from $[0,1]$ to $\mathbb{R}$. We endow $C([0,1])$ with the L^∞ norm defined by

$$||f||_\infty = \sup\{|f(x)|\ :\ 0 \le x \le 1\}.$$

(a) Show that $\|T\| \le 1$.
(b) If $g \in C([0,1])$ is defined by $g(x) = 1$, find $|T(g)|$, and use this to compute $\|T\|$.

SOLUTIONS:

(a) Let $f \in C([0,1])$, and let $M = ||f||_\infty$. Then

$$|T(f)| = \left|\int_0^1 f(x)\ dx\right| \le \int_0^1 |f(x)|\ dx \le \int_0^1 M\ dx = M = 1 \cdot ||f||_\infty.$$

Therefore $||T|| \le 1$.

(b) We have that

$$|T(g)| = \left|\int_0^1 g(x)\ dx\right| = \left|\int_0^1 1\ dx\right| = 1 = 1 \cdot ||g||_\infty.$$

This together with (a) implies that $||T|| = 1$.

Fall 2022 #5. Define

$$\ell^2(\mathbb{N};\mathbb{R}) := \left\{ (x_n)_{n=1}^\infty \ \middle|\ x_1, x_2, x_3, \dots \in \mathbb{R} \text{ and } \sum_{n=1}^\infty x_n^2 < \infty \right\}.$$

In other words, $\ell^2(\mathbb{N};\mathbb{R})$ is the set of all "square-summable" sequences of real numbers. Recall that $\ell^2(\mathbb{N};\mathbb{R})$ is an inner product space with the inner product

$$\langle (x_n)_{n=1}^{\infty}, (y_n)_{n=1}^{\infty} \rangle = \sum_{n=1}^{\infty} x_n y_n.$$

(You may assume without proof that this defines an inner product.)

Define

$$r_n = \begin{cases} 2^{-(n+3)/4} & \text{if } n \text{ is odd} \\ 2^{-(n+2)/4} & \text{if } n \text{ is even} \end{cases}$$

and $s_n = (-1)^n r_n$. So

$$(r_n)_{n=1}^{\infty} = \left(2^{-1}, 2^{-1}, 2^{-3/2}, 2^{-3/2}, 2^{-2}, 2^{-2}, 2^{-5/2}, 2^{-5/2}, \dots\right), \text{ and}$$
$$(s_n)_{n=1}^{\infty} = \left(2^{-1}, -2^{-1}, 2^{-3/2}, -2^{-3/2}, 2^{-2}, -2^{-2}, 2^{-5/2}, -2^{-5/2}, \dots\right).$$

(a) Prove that $(r_n)_{n=1}^{\infty} \in \ell^2(\mathbb{N};\mathbb{R})$ and $(s_n)_{n=1}^{\infty} \in \ell^2(\mathbb{N};\mathbb{R})$. (Recall the geometric series formula: If x is a real number such that $-1 < x < 1$, then $\sum_{n=1}^{\infty} x^n = \frac{x}{1-x}$.)

(b) Prove that $\{(r_n)_{n=1}^{\infty}, (s_n)_{n=1}^{\infty}\}$ is an orthonormal family in $\ell^2(\mathbb{N};\mathbb{R})$.

(c) Find real numbers a and b so that the quantity $J(a,b)$ below is as small as possible.

$$J(a,b) = \sum_{n=1}^{\infty} \left(2^{-n} - a r_n - b s_n\right)^2.$$

SOLUTIONS:

(a) We have that

$$\sum_{n=1}^{\infty} r_n^2 = 2^{-2} + 2^{-2} + 2^{-3} + 2^{-3} + 2^{-4} + 2^{-4} + 2^{-5} + 2^{-5} + \cdots$$
$$= 2(2^{-2}) + 2(2^{-3}) + 2(2^{-4}) + 2(2^{-5}) + \cdots$$
$$= 2^{-1} + 2^{-2} + 2^{-3} + 2^{-4} + \cdots$$
$$= \sum_{n=1}^{\infty} (1/2)^n$$
$$= (1/2)/(1 - 1/2) = 1 < \infty$$

by the geometric series formula. So $(r_n)_{n=1}^{\infty} \in \ell^2(\mathbb{N}; \mathbb{R})$.

Similarly we find that

$$\sum_{n=1}^{\infty} s_n^2 = 1.$$

So $(s_n)_{n=1}^{\infty} \in \ell^2(\mathbb{N}; \mathbb{R})$.

(b) We have that

$$||(r_n)_{n=1}^{\infty}|| = \sqrt{\langle (r_n)_{n=1}^{\infty}, (r_n)_{n=1}^{\infty} \rangle} = \sqrt{\sum_{n=1}^{\infty} r_n^2} = \sqrt{1} = 1$$

by the computation in (a). Similarly

$$||(s_n)_{n=1}^{\infty}||.$$

Finally,

$$\langle (r_n)_{n=1}^{\infty}, (s_n)_{n=1}^{\infty} \rangle = \sum_{n=1}^{\infty} r_n s_n$$
$$= (2^{-1})^2 - (2^{-1})^2 + (2^{-3/2})^2 - (2^{-3/2})^2 + (2^{-2})^2 - (2^{-2})^2 + \cdots = 0.$$

(c) Let $t_n = 2^{-n}$. Observe that

$$J(a, b) = ||(t_n)_{n=1}^{\infty} - a(r_n)_{n=1}^{\infty} - b(s_n)_{n=1}^{\infty}||^2.$$

(This is legitimate because $\sum_{n=1}^{\infty} 2^{-2n} < \infty$, which implies that $(t_n)_{n=1}^{\infty} \in \ell^2(\mathbb{N};\mathbb{R})$.)

So we are really trying to do is minimize the ℓ^2 distance from $(t_n)_{n=1}^{\infty}$ to M, where

$$M = \operatorname{span}\{(r_n)_{n=1}^{\infty}, (s_n)_{n=1}^{\infty}\}$$

This distance is minimized when we take a and b to be the coefficients of the projection of $(t_n)_{n=1}^{\infty}$ onto M. Thus take:

$$a = \langle (t_n)_{n=1}^{\infty}, (r_n)_{n=1}^{\infty} \rangle$$

$$= 2^{-1}(2^{-1}) + 2^{-2}(2^{-1}) + 2^{-3}(2^{-3/2}) + 2^{-4}(2^{-3/2}) + 2^{-5}(2^{-2}) + 2^{-6}(2^{-2}) + \cdots$$

$$= [2^{-1}(2^{-1}) + 2^{-3}(2^{-3/2}) + 2^{-5}(2^{-2}) + \cdots] + [2^{-2}(2^{-1}) + 2^{-4}(2^{-3/2}) + 2^{-6}(2^{-2}) + \cdots]$$

$$= [2^{-2} + 2^{-9/2} + 2^{-7} + \cdots] + [2^{-3} + 2^{-11/2} + 2^{-8} + \cdots]$$

$$= 2^{1/2}\sum_{n=1}^{\infty}(2^{-5/2})^n + 2^{-1/2}\sum_{n=1}^{\infty}(2^{-5/2})^n$$

$$= 2^{1/2} \cdot \frac{2^{-5/2}}{1-2^{-5/2}} + 2^{-1/2} \cdot \frac{2^{-5/2}}{1-2^{-5/2}} = \frac{2^{-2}+2^{-3}}{1-2^{-5/2}} = \frac{2+1}{2^3-2^{1/2}} = \frac{3}{8-\sqrt{2}}.$$

Similarly take:

$$b = \langle (t_n)_{n=1}^{\infty}, (s_n)_{n=1}^{\infty} \rangle$$

$$= 2^{-1}(2^{-1}) - 2^{-2}(2^{-1}) + 2^{-3}(2^{-3/2}) - 2^{-4}(2^{-3/2}) + 2^{-5}(2^{-2}) - 2^{-6}(2^{-2}) + \cdots$$

$$= [2^{-1}(2^{-1}) + 2^{-3}(2^{-3/2}) + 2^{-5}(2^{-2}) + \cdots] - [2^{-2}(2^{-1}) + 2^{-4}(2^{-3/2}) + 2^{-6}(2^{-2}) + \cdots]$$

$$= [2^{-2} + 2^{-9/2} + 2^{-7} + \cdots] - [2^{-3} + 2^{-11/2} + 2^{-8} + \cdots]$$

$$= 2^{1/2}\sum_{n=1}^{\infty}(2^{-5/2})^n - 2^{-1/2}\sum_{n=1}^{\infty}(2^{-5/2})^n$$

$$= 2^{1/2} \cdot \frac{2^{-5/2}}{1-2^{-5/2}} - 2^{-1/2} \cdot \frac{2^{-5/2}}{1-2^{-5/2}} = \frac{2^{-2}-2^{-3}}{1-2^{-5/2}} = \frac{2-1}{2^3-2^{1/2}} = \frac{1}{8-\sqrt{2}}$$

Fall 2022 #6. Let V be a normed vector space over $\mathbb{F}$, where $\mathbb{F}$ may be either the field of real numbers or the field of complex numbers. Let A be a linear subspace of V. Let C be the set of all $x \in V$ such that there is a sequence $\{x_n\}_{n=1}^{\infty}$ in A converging to x. (In other words, C is the closure of A.) Prove that C is a linear subspace of V.

Proof. Note that $C \subset V$ by definition of C.

First we show that $0 \in C$. We know that $0 \in A$ because A is a linear subspace of V. The constant sequence $\{0\}$ converges to 0. Thus $0 \in C$.

Next, let $x, y \in C$. We will show that $x+y \in C$. By definition of C, we know that there are sequences $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ in A converging to x and y, respectively. Because A is a linear subspace of V, we know that $x_n+y_n \in A$ for all n. The sequence $\{x_n+y_n\}_{n=1}^{\infty}$ is then a sequence in A converging to $x+y$. Hence $x+y \in C$.

Finally, let $\lambda \in \mathbb{F}$ and $x \in C$. By definition of C, we know that there is a sequence $\{x_n\}_{n=1}^{\infty}$ in A converging to x. Because A is a linear subspace of V, we know that $\lambda x_n \in A$ for all n. The sequence $\{\lambda x_n\}_{n=1}^{\infty}$ is then a sequence in A converging to λx. Hence $\lambda x \in C$.

We have shown that $C \subset V$, that C contains the zero vector, that C is closed under addition, and that C is closed under scalar multiplication. Therefore C is a linear subspace of V. □

Fall 2022 #7. Let $f(x) = x(\pi - x)$ for $x \in (0, \pi)$.

(a) Extend the function f to the interval $(-\pi, \pi)$ such that it becomes an odd function. Please write down the expression of the extended function $F(x)$ on $(-\pi, \pi)$.

(b) We extend $F(x)$ from Part (a) to be 2π-periodic on $\mathbb{R}$. Find the Fourier series for $F(x)$ in the trigonometric form.

(c) Use the result of Part (b) to find the value of the infinite series

$$1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \frac{1}{9^3} - \cdots$$

SOLUTIONS:

(a) What we want is $F(x) = f(x)$ on $(0, \pi)$, and $F(-x) = -F(x)$ for $x \in (-\pi, \pi)$:

$$F(x) = \begin{cases} x(\pi - x) & \text{on } [0, \pi) \\ x(\pi + x) & \text{on } (-\pi, 0). \end{cases}$$

(b) Since F is odd, the trigonometric Fourier series is

$$F(x) \sim \sum_{k=1}^{\infty} b_k \sin(kx), \quad b_k = \frac{2}{\pi} \int_0^{\pi} F(t) \sin(kt)\, dt.$$

(Different books use slightly different definitions for Fourier series, so your answer may differ slightly by some constants. The answer to (c) will work out the same no matter what, though.)

By careful calculations, we obtain

$$b_k = \frac{4}{\pi k^3} - \frac{4}{\pi k^3} \cos(k\pi),$$

and thus,

$$b_k = \begin{cases} 0, & \text{if } k = 2n, \\ \frac{8}{\pi(2n-1)^3}, & \text{if } k = 2n - 1, \end{cases}$$

for $n = 1, 2, 3, \ldots$

Therefore,

$$F(x) \sim \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{\sin[(2n-1)x]}{(2n-1)^3}.$$

(c) Note that F is differentiable at $\pi/2$. Therefore the Fourier series for F converges to $F(\pi/2)$ at $x = \pi/2$.

If $x = \pi/2$, by Part (a), we have

$$F(\pi/2) = \frac{\pi^2}{4}.$$

On the other side, by Part (b), we have

$$F(\pi/2) = \frac{8}{\pi}\left(\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \cdots\right).$$

Therefore,

$$1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \frac{1}{9^3} - \cdots = \frac{\pi^3}{32}.$$