THE GAME CREATION OPERATOR

Joint work with Urban Larsson and Matthieu Dufour

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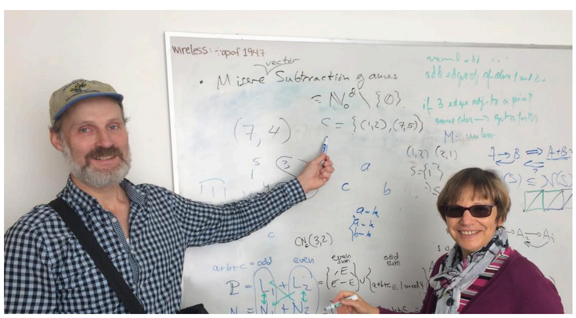
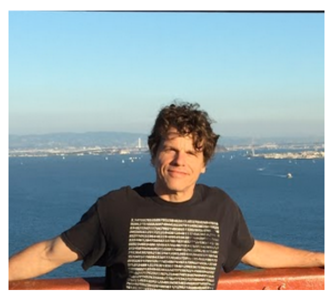




Much of this work was done during my sabbatical visit to Berkeley in October 2016

Outline

- Basic background on combinatorial games
- Definition of subtraction games
- Some examples: Nim, Wythoff
- History of the game creation operator
- Our results
- Future work

The Basics

- A two-player game is called a **combinatorial game** if there is no randomness involved and all possible moves are known to each player.
- A combinatorial game is called **impartial** if both players have the same allowed moves
- **Examples:**

- Under **normal play**, the last player to move wins. Under **misère play**, the last player to move loses.

Main Question:

Who wins in a combinatorial game from a specific position, assuming both players play optimally?

Subtraction Games

- A **subtraction** or **take-away game** is played on one or more stacks of tokens
- **Positions** are described as vectors of stack heights
- The **subtraction set M** consists of the possible moves in the form of subtraction vectors. A move can be used as long as it does not result in negative stack height(s)

(5, 3, 2, 1)

Take one token from stack 1

$\leftrightarrow$

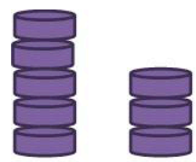
(1, 0, 0, 0)

Subtraction Sets

Examples:

- **NIM** on **one** stack   **M = {1, 2, 3, …}**

- **WYTHOFF** is played on two stacks. Can either take
  - one or more tokens from **one** stack, or
  - the **same** number from **both** stacks.



M = {(1,0), (2,0),… , (0,1), (0,2),… , (1,1),(2,2),…}

Impartial Games

Only two possible outcome classes:

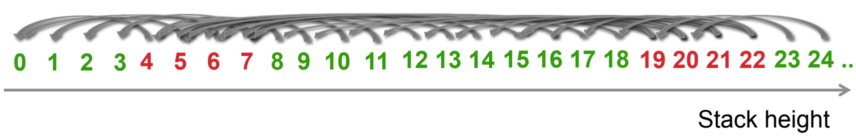
- **Losing** positions
- **Winning** positions

Characterization of positions

- From a **losing** position, **all** allowed moves lead to a **winning** position
- From a **winning** position, there is **at least one** move to a **losing** position.
- In misère play, the **terminal** positions are **winning** positions

Recursive Determination of Outcome Class

- Game **M**= {4, 7, 11}
- We will color **winning** and **losing** positions
- The terminal positions are 0, 1, 2, 3
- Pattern that emerges is an alternating sequence of **4 losing** positions followed by **11 winning** positions (after the terminal positions in the beginning)



★-Operator

Observation: For subtraction games, positions and allowed moves have the **same structure**! This allows us to iteratively create new games.

The **★-operator** is defined as follows:

- We start with a subtraction game **M** that is described by the allowed moves.
- We compute the set of losing positions, **L(M)**
- The losing positions of **M** become the moves for the game **M**★

- Notation: $\mathbf{M}^0 = \mathbf{M}$, $\mathbf{M}^n = (\mathbf{M}^{n-1})^\star$
- **M** is **reflexive** if **M** = **M**★

How did the ★-Operator come about?

WYTHOFF

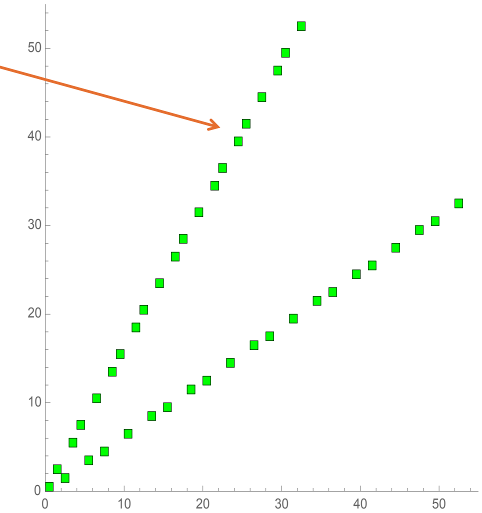
- The losing positions of **WYTHOFF** (under normal play) are closely related to the golden ratio $\varphi = \frac{1+\sqrt{5}}{2}$:

$$\mathcal{L} = \{(\lfloor n \cdot \varphi \rfloor, \lfloor n \cdot \varphi \rfloor + n) | n \geq 0\}$$

- We only list positions of the form (x,y), but by symmetry, (y,x) is also a losing position.

Visualization of the Losing Positions

a_n	b_n
0	0
1	2
3	5
4	7
6	10
8	13
9	15
11	18
12	20
14	23
16	26
17	28



Recursive Creation of the Losing Positions

- The losing positions can also be computed recursively.

n	0	1	2	3	4	5	6	7
a_n	0	1	3	4	6	8	9	11
b_n	0	2	5	7	10	13	15	18

- Let a_n = the smallest non-negative integer not yet used and set $b_n = a_n + n$. Repeat.
- By creation, sequences $\{a_n\}$ and $\{b_n\}$ are complementary, in fact, they are homogenous Beatty sequences.

Complementary Beatty Sequences

From American Mathematical Monthly, **33** (3): 159

1926] PROBLEMS AND SOLUTIONS 159

PROBLEMS AND SOLUTIONS

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND H. L. OLSON.

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PROBLEMS FOR SOLUTION

(N.B. Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks, or results found in readily accessible sources, will not be proposed as problems for solution in the MONTHLY. In so far as possible, however, the editors will be glad to assist members of the Association with their difficulties in the solution of such problems.)

3173. Proposed by Samuel Beatty, University of Toronto.

If X is a positive irrational number and Y its reciprocal, prove that the sequences

$$(1+X)\,,\quad 2(1+X)\,,\quad 3(1+X)\,,\cdots$$
$$(1+Y)\,,\quad 2(1+Y)\,,\quad 3(1+Y)\,,\cdots$$

contain one and only one number between each pair of consecutive positive integers.

Complementary Beatty Sequences & Games

Duchêne-Rigo Conjecture: Every complementary pair of homogeneous Beatty sequences forms the set of losing positions for some invariant impartial game.

This conjecture was proved by Larsson, Hegarty and Fraenkel using the game creation operator (★-operator)

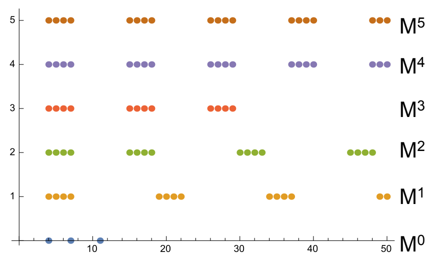
Back to the ★-Operator

Questions for Misère-Play ★-Operator

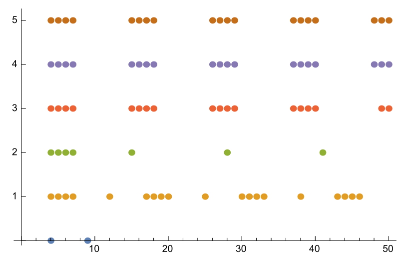
- **Question 1**: Does the misère-play ★-operator converge (point-wise)?
- **Question 2**: What feature(s) of **M** determines the limit game for its sequence?
- **Question 3**: Limit games are (by definition) reflexive. What is the structure of reflexive games and/or limit games (if they exist)?
- **Question 4**: How quickly does convergence occur?

Example for One Stack

★-operator applied five times to initial game



$M^0 = \{4, 7, 11\}$



$G^0 = \{4, 9\}$

Observations from Example

- Looks like there is convergence (fixed point) for each of the games
- Limit games seem to have a periodic structure: blocks of moves alternate with blocks of non-moves
- $M^0 = \{4, 7, 11\}$ and $G^0 = \{4, 9\}$ seem to have the same limit game

Question: What do the two sets M^0 and G^0 have in common?

Answer: The minimal element, **$k = 4$**.

Q1: Convergence Result

Theorem
Starting from any game **M** on *d* stacks, the sequence of games created by the misère-play ★-operator converges to a (reflexive) limit game $\mathbf{M}^{\infty}$.

Convergence Result

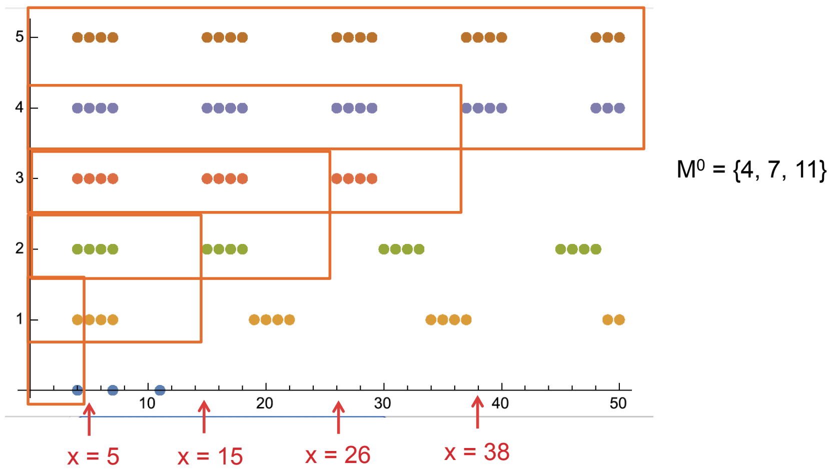
Proof idea: (for *d* stacks)

- Positions become fixed either as moves or non-moves from “smaller to larger”. There are four possibilities:

	move in $\mathbf{M}^{i+1}$	Non-move in $\mathbf{M}^{i+1}$
move in $\mathbf{M}^{i}$	Fixed as a move	Erased as move
Non-move in $\mathbf{M}^{i}$	Introduced as move	Fixed as non-move

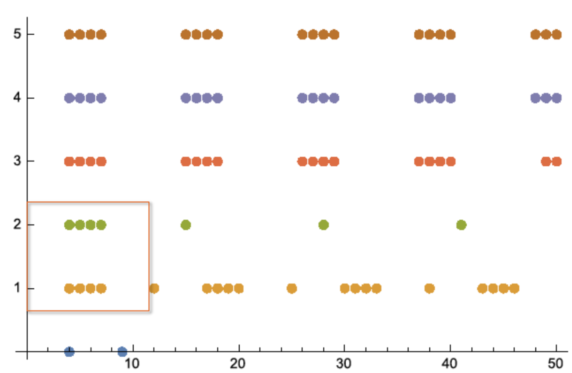
- Show that smallest element not yet fixed becomes fixed.

Proof by Picture



Proof by Picture

- Not all positions switch from non-move to move:



Q2: Which Feature of **M** Determines **M**$^\infty$?

Theorem

Two games **M** and **G** (played on the same number of stacks) have the same limit game if and only if their unique **sets of minimal elements** (with the usual partial order) are the same.

Q3: Characteristic of Reflexive Games

- The following result is somewhat technical, but it is a general result for games on any number of stacks.
- It is used to prove specific results for one and two stacks.

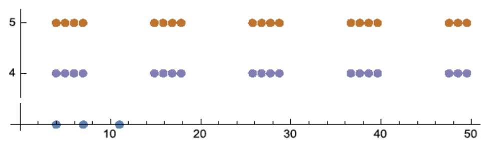
Theorem

The game **A** on *d* stacks is reflexive if and only if its set of moves **A** (as a set) satisfies

$$\mathbf{A} + \mathbf{A} = \mathbf{A}^c \setminus T_{\mathbf{A}}$$

where $T_{\mathbf{A}}$ is the set of terminal positions of the game **A**.

Structure of Reflexive Games on One Stack



Pattern:
- period 3k-1;
- starts at k
- has k moves, followed by 2k-1 non-moves.

$\mathbf{M_k} := \{ i\, p_k + k, \ldots, i\, p_k + (2k-1) \mid i = 0,1,\ldots\}$, where $p_k = 3k-1$

Theorem
The game **M** is reflexive iff **M** = $\mathbf{M}_k$ for some k > 0.

Structure of Limit/Reflexive Games on Two Stacks

Classification of games according to minimal moves

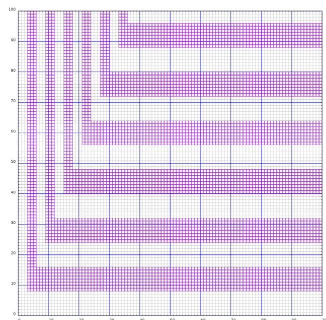
① **Exactly one minimal move**
  a. Not on an axis
  b. On one of the axes

② Exactly two minimal moves
  a. No minimal move on an axis
  b. Exactly one move is on an axis
  c. **Both moves are on the axes**

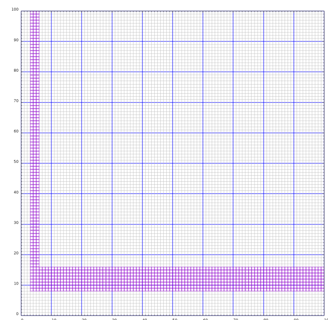
③ Three or more minimal moves
  a. **No minimal move on an axis**
  b. Exactly one move is on an axis
  c. **Two moves are on the axes**

Exactly one minimum not on axis

Example: **M** = {(3,8)}



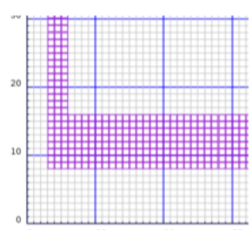
M^1



$M^2 = M^3$

Exactly one minimum not on axis

M= {(3,8)}

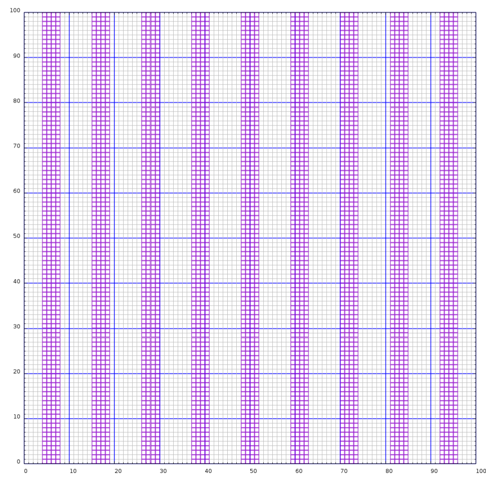


Anchored at (k,l) = (3,8)
Vertical width is l = 8
Horizontal width is k = 3

Theorem [Bloomfield]
If min(M) = {(k,l)}, then the limit game is reached in two steps, and is in the form of an L anchored at (k,l) with vertical width l and horizontal width k.

Exactly one minimum on axis

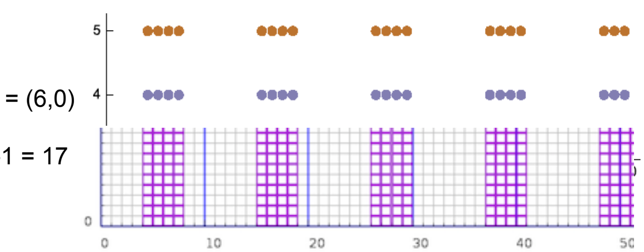
Example: **M**={4,0}; shown is sequence M^1 to $M^5 = M^6$



Exactly one minimum on axis

M = {(4,0)}

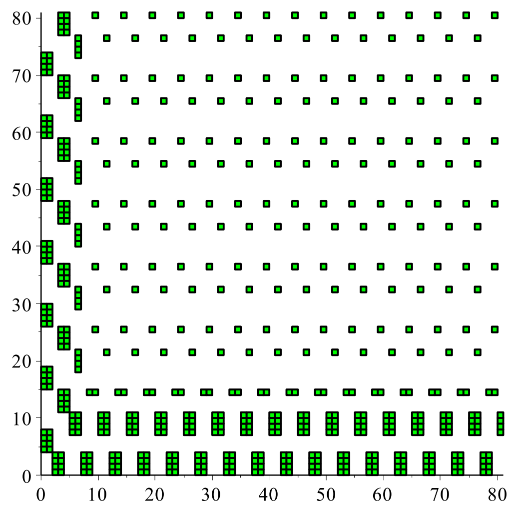
Anchored at (k,l) = (6,0)
Bands of width k
Period = p_k = 3k-1 = 17



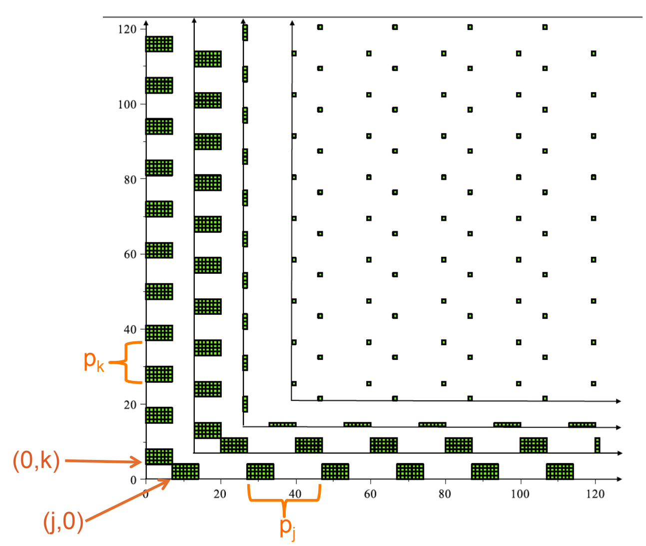
This case inherits its structure from the one-stack case.

Theorem [Bloomfield]
If min(M) = {(k,0)} the limit game has the form of vertical bands of width k, with a period of p_k, so M^∞ ={(x,y), x in M_k}.

Example: Two Minima on Axes



Definition of Game $M_{j,k}$



Reflexivity of $M_{j,k}$

Theorem [Bloomfield, Dufour, Heubach, Larsson]
The game $\mathbf{M_{j,k}}$ is reflexive.

Corollary
The limit game of a set **M** equals the game $\mathbf{M_{j,k}}$ if and only if the **set of minimal elements** of **M** is {**(j,0),(0,k)**}.

Q4: How Long until Convergence?

- We can only answer this question for specific initial games

Theorem
For **M = {*k*}** with k > 1 it takes exactly **5** iterations for the limit game to appear for the first time.

Corollary
For **M = {(k,0}}** or **M = {(0,l}}** with k,l > 1 it takes exactly **5** iterations for the limit game to appear for the first time.

Proof: We explicitly derive the games M^1 through M^5.

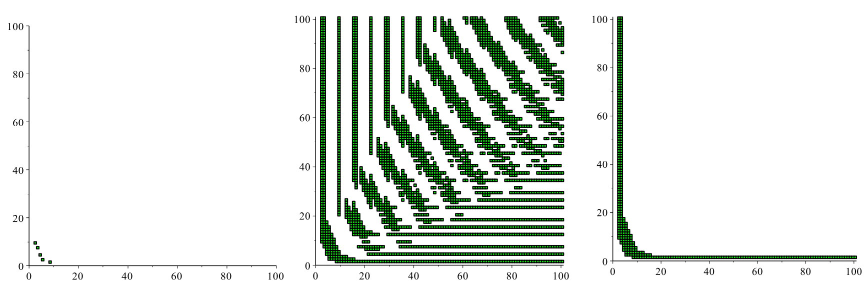
- For other games on two stacks we have very varied results from our computer explorations

Future Work

1. Investigate the structure of the limit games in the other classes for games on two stacks
2. Computer experiments for three minimal elements have produced "L-shaped" limit games, limit games with diagonal stripes, and limit games that combine the two features

Three+ Minimal Moves – None on Axis

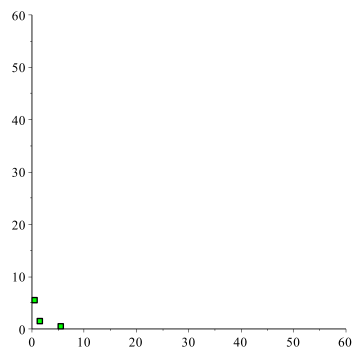
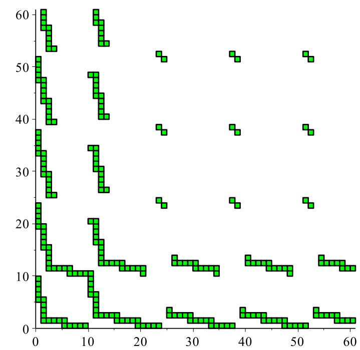
M = {(2,9), (3,7), (4,4), (5,2), (8,1)}



Convergence after 2 steps!

Three Minimal Moves – Two on Axes

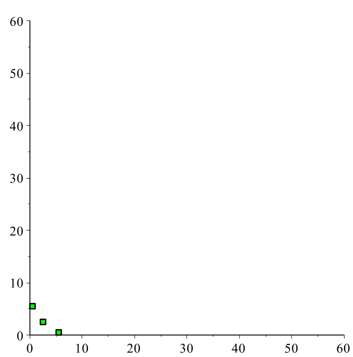
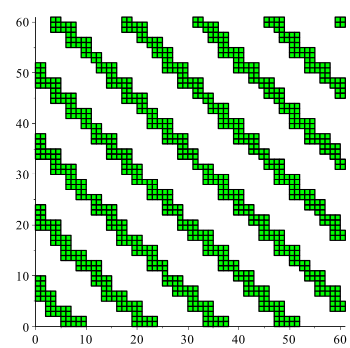
M = {(0,5), (1,1), (5,0)}

Convergence after 8 steps!

Three minimal moves – two on axes

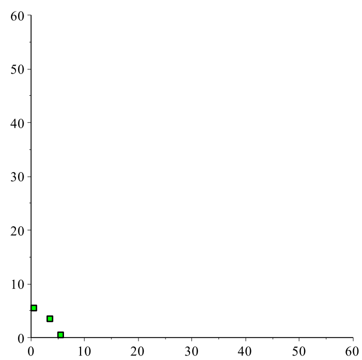
M = {(0,5), (2,2), (5,0)}

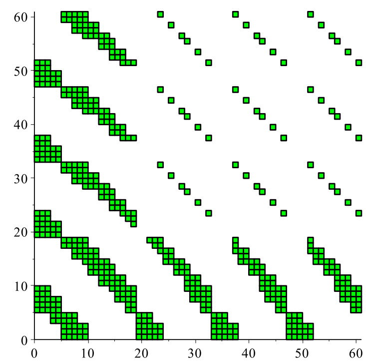



Convergence after 7 steps!

Three Minimal Moves – Two on Axes

M = {(0,5), (3,3), (5,0)}

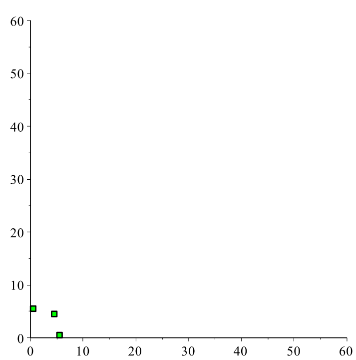


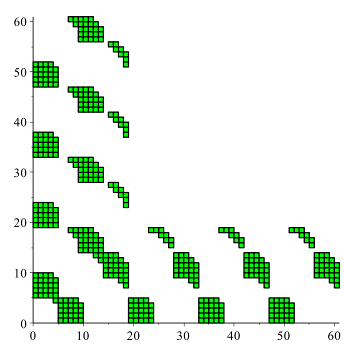


Convergence after 7 steps!

Three Minimal Moves – Two on Axes

M = {(0,5), (4,4), (5,0)}





Convergence after 6 steps!

Future Work

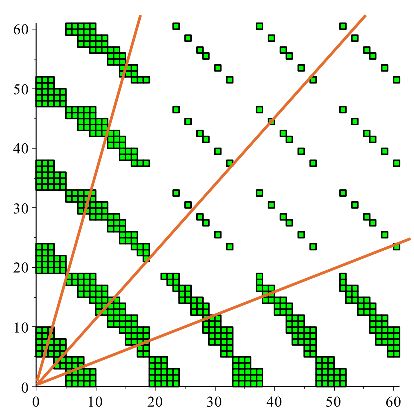
1. Investigate the structure of the limit games in the other classes for games on two stacks

2. We have observed "L-shaped" limit games, limit games with diagonal stripes, and limit games that combine the two features

3. Number of steps to convergence, or showing that it happens in a finite number of steps for all games or for games of a particular (sub-) class

Future Work

Conjecture
For all subtraction games on two stacks, limit games under the misère *-operator are ultimately periodic along any line of rational slope.



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- U. Larsson, P. Hegarty, and A. S. Fraenkel. Invariant and dual subtraction games resolving the Duchêne-Rigo conjecture, *Theoretical Computer Science*, 412, pp 729-735, 2011.
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THANK YOU!

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Slides will be posted at

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