Avoidance of partially ordered patterns in compositions

S. Heubach[1] S. Kitaev[2] T. Mansour[3]

[1]Dept. of Mathematics, California State Univ. Los Angeles

[2]Department of Mathematics, Reykjavik University

[3]Department of Mathematics, Haifa University

August 27, 2008
Math Colloquium, Dalhousie University, Halifax

Outline

Outline

Outline

Outline

Prior Work

- **Permutations** avoiding a **permutation pattern**
- **Permutations** avoiding general patterns or set of patterns
- Words avoiding **general patterns** or **set of patterns**
- **Compositions** enumerated according to **rises, levels and drops** (= 2-letter patterns)
- **Compositions** avoiding **3-letter patterns**
- **Compositions** enumerated according to **segmented partially ordered (generalized) patterns** = POPs

$\Rightarrow$ **Compositions** avoiding **POPs**

Prior Work

- **Permutations** avoiding a **permutation pattern**
- **Permutations** avoiding general patterns or set of patterns
- Words avoiding **general patterns** or **set of patterns**
- **Compositions** enumerated according to **rises, levels and drops** (= 2-letter patterns)
- **Compositions** avoiding **3-letter patterns**
- **Compositions** enumerated according to **segmented partially ordered (generalized) patterns** = POPs

$\Rightarrow$ **Compositions** avoiding **POPs**

Prior Work

- **Permutations** avoiding a **permutation pattern**
- **Permutations** avoiding general patterns or set of patterns
- Words avoiding **general patterns** or **set of patterns**
- **Compositions** enumerated according to **rises, levels and drops** (= 2-letter patterns)
- **Compositions** avoiding **3-letter patterns**
- **Compositions** enumerated according to **segmented partially ordered (generalized) patterns** = POPs

$\Rightarrow$ **Compositions** avoiding **POPs**

Prior Work

- **Permutations** avoiding a **permutation pattern**
- **Permutations** avoiding general patterns or set of patterns
- Words avoiding **general patterns** or **set of patterns**
- **Compositions** enumerated according to **rises, levels and drops** (= 2-letter patterns)
- **Compositions** avoiding **3-letter patterns**
- **Compositions** enumerated according to **segmented partially ordered (generalized) patterns** = POPs

$\Rightarrow$ **Compositions** avoiding **POPs**

Prior Work

- **Permutations** avoiding a **permutation pattern**
- **Permutations** avoiding general patterns or set of patterns
- Words avoiding **general patterns** or **set of patterns**
- **Compositions** enumerated according to **rises, levels and drops** (= 2-letter patterns)
- **Compositions** avoiding **3-letter patterns**
- **Compositions** enumerated according to **segmented partially ordered (generalized) patterns** = POPs

$\Rightarrow$ **Compositions** avoiding **POPs**

Prior Work

- **Permutations** avoiding a **permutation pattern**
- **Permutations** avoiding general patterns or set of patterns
- Words avoiding **general patterns** or **set of patterns**
- **Compositions** enumerated according to **rises, levels and drops** (= 2-letter patterns)
- **Compositions** avoiding **3-letter patterns**
- **Compositions** enumerated according to **segmented partially ordered (generalized) patterns** = POPs

$\Rightarrow$ **Compositions** avoiding **POPs**

Prior Work

- **Permutations** avoiding a **permutation pattern**
- **Permutations** avoiding general patterns or set of patterns
- Words avoiding **general patterns** or **set of patterns**
- **Compositions** enumerated according to **rises, levels and drops** (= 2-letter patterns)
- **Compositions** avoiding **3-letter patterns**
- **Compositions** enumerated according to **segmented partially ordered (generalized) patterns** = POPs

  $\Rightarrow$ **Compositions** avoiding **POPs**

Compositions

Definition

Let $A = \{a_1, a_2, \ldots, a_k\}$ be an ordered subset of $\mathbb{N}$. A composition of n with m parts in A is an ordered sequence $\sigma = \sigma_1\sigma_2\ldots\sigma_m$ with $\sum_{i=1}^{m} \sigma_i = n$ and $\sigma_i \in A$. We denote the set of all compositions of n with (m) parts in A by C_n^A ($C_{n;m}^A$).

Example

The compositions of 4 (with parts in $\mathbb{N}$) are 4, 31, 13, 22, 211, 121, 112, 1111.

Compositions

Definition

Let $A = \{a_1, a_2, \ldots, a_k\}$ be an ordered subset of $\mathbb{N}$. A composition of n with m parts in A is an ordered sequence $\sigma = \sigma_1\sigma_2\ldots\sigma_m$ with $\sum_{i=1}^{m}\sigma_i = n$ and $\sigma_i \in A$. We denote the set of all compositions of n with (m) parts in A by C_n^A $(C_{n;m}^A)$.

Example

The compositions of 4 (with parts in $\mathbb{N}$) are 4, 31, 13, 22, 211, 121, 112, 1111.

Patterns

Definition

Let $[k] = \{1, 2, \ldots, k\}$. Then the elements in $[k]^n$ are called words of length n over $[k]$. A generalized pattern τ is a word in $[\ell]^k$ that contains each letter from $[\ell]$, possibly with repetitions and dashes.

- pattern with no adjacency requirement = classical pattern
- pattern with no dashes = consecutive or segmented pattern

1234 **1-23-4** **1-2-3-4**

Reduced sequence

Definition

For any sequence $\sigma = \sigma_1\sigma_2\ldots\sigma_m$, we define its reduced form to be the sequence $s_1s_2\cdots s_m$, where $s_i = \ell$ if the σ_i is ℓ-th smallest term.

The reduced form just takes into account the relative size of the sequence terms, and maps the sequence to the set $[k]$, where k is the number of distinct terms in the sequence.

Example

The reduced form of the sequence 35237 is 23124, since the terms of the sequence are in order $2 < 3 = 3 < 5 < 7$.

Reduced sequence

Definition

For any sequence $\sigma = \sigma_1\sigma_2 \ldots \sigma_m$, we define its reduced form to be the sequence $s_1s_2 \cdots s_m$, where $s_i = \ell$ if the σ_i is ℓ-th smallest term.

The reduced form just takes into account the relative size of the sequence terms, and maps the sequence to the set $[k]$, where k is the number of distinct terms in the sequence.

Example

The reduced form of the sequence 35237 is 23124, since the terms of the sequence are in order $2 < 3 = 3 < 5 < 7$.

Reduced sequence

Definition

For any sequence $\sigma = \sigma_1\sigma_2\ldots\sigma_m$, we define its reduced form to be the sequence $s_1s_2\cdots s_m$, where $s_i = \ell$ if the σ_i is ℓ-th smallest term.

The reduced form just takes into account the relative size of the sequence terms, and maps the sequence to the set $[k]$, where k is the number of distinct terms in the sequence.

Example

The reduced form of the sequence 35237 is 23124, since the terms of the sequence are in order $2 < 3 = 3 < 5 < 7$.

Containing and Avoiding Patterns

Definition

A composition σ contains a pattern $\tau = \tau_1\tau_2\ldots\tau_k$ if the reduced form of any subsequence of k consecutive terms of σ equals τ. Otherwise we say that σ avoids the subword pattern τ or is τ-avoiding and write $\sigma \in C_n^A(\tau)$ $(\sigma \in C_{n;m}^A(\tau))$.

Example

241874 avoids 312

241874 contains five occurrences of 1-32:

Containing and Avoiding Patterns

Definition

A composition σ contains a pattern $\tau = \tau_1\tau_2\ldots\tau_k$ if the reduced form of any subsequence of k consecutive terms of σ equals τ. Otherwise we say that σ avoids the subword pattern τ or is τ-avoiding and write $\sigma \in C_n^A(\tau)$ ($\sigma \in C_{n;m}^A(\tau)$).

Example

241874 avoids **312**

241874 contains five occurrences of **1-32**:

Containing and Avoiding Patterns

Definition

A composition σ contains a pattern $\tau = \tau_1\tau_2\ldots\tau_k$ if the reduced form of any subsequence of k consecutive terms of σ equals τ. Otherwise we say that σ avoids the subword pattern τ or is τ-avoiding and write $\sigma \in C_n^A(\tau)$ ($\sigma \in C_{n;m}^A(\tau)$).

Example

241874 avoids **312**

241874 contains five occurrences of **1-32**:

Containing and Avoiding Patterns

Definition

A composition σ contains a pattern $\tau = \tau_1\tau_2\ldots\tau_k$ if the reduced form of any subsequence of k consecutive terms of σ equals τ. Otherwise we say that σ avoids the subword pattern τ or is τ-avoiding and write $\sigma \in C_n^A(\tau)$ $(\sigma \in C_{n;m}^A(\tau))$.

Example

241874 avoids **312**

241874 contains five occurrences of **1-32**:

241874

Containing and Avoiding Patterns

Definition

A composition σ contains a pattern $\tau = \tau_1\tau_2\ldots\tau_k$ if the reduced form of any subsequence of k consecutive terms of σ equals τ. Otherwise we say that σ avoids the subword pattern τ or is τ-avoiding and write $\sigma \in C_n^A(\tau)$ $(\sigma \in C_{n;m}^A(\tau))$.

Example

241874 avoids **312**

241874 contains five occurrences of **1-32**:

241874 **241874**

Containing and Avoiding Patterns

Definition

A composition σ contains a pattern $\tau = \tau_1\tau_2\ldots\tau_k$ if the reduced form of any subsequence of k consecutive terms of σ equals τ. Otherwise we say that σ avoids the subword pattern τ or is τ-avoiding and write $\sigma \in C_n^A(\tau)$ $(\sigma \in C_{n;m}^A(\tau))$.

Example

241874 avoids **312**

241874 contains five occurrences of **1-32**:

241874 **241874** **241874**

Containing and Avoiding Patterns

Definition

A composition σ contains a pattern $\tau = \tau_1\tau_2\ldots\tau_k$ if the reduced form of any subsequence of k consecutive terms of σ equals τ. Otherwise we say that σ avoids the subword pattern τ or is τ-avoiding and write $\sigma \in C_n^A(\tau)$ $(\sigma \in C_{n;m}^A(\tau))$.

Example

241874 avoids **312**

241874 contains five occurrences of **1-32**:

241874 **241874** **241874** **241874**

Containing and Avoiding Patterns

Definition

A composition σ contains a pattern $\tau = \tau_1\tau_2\ldots\tau_k$ if the reduced form of any subsequence of k consecutive terms of σ equals τ. Otherwise we say that σ avoids the subword pattern τ or is τ-avoiding and write $\sigma \in C_n^A(\tau)$ ($\sigma \in C_{n;m}^A(\tau)$).

Example

241874 avoids **312**

241874 contains five occurrences of **1-32**:

241874 **241874** **241874** **241874** **241874**

Partially Ordered Patterns

Definition

A partially ordered pattern POP τ is a word consisting of letters from a partially ordered alphabet $\mathcal{T}$.

- If letters *a* and *b* are incomparable in a POP τ, then the relative size of the letters in σ corresponding to *a* and *b* is unimportant in an occurrence of τ in σ.
- Comparable letters have the same number of primes.
- Letters without primes are considered to be comparable to all other letters.

Partially Ordered Patterns

Example

Let $\mathcal{T} = \{1', 1'', 2''\}$ with the only relation $1'' < 2''$. Then **113425** contains three occurrences of **1′1″2″** and seven occurrences of **1′-1″2″**

- 113425, 113425, 113425
- 113425, 113425, 113425, 113425

- Avoidance of POPs $\leftrightarrow$ multi-avoidance of a set of patterns: avoiding 2′-1-2″ $\leftrightarrow$ simultaneously avoiding {2-1-2, 3-1-2, 2-1-3}.

Partially Ordered Patterns

Example

Let $\mathcal{T} = \{1', 1'', 2''\}$ with the only relation $1'' < 2''$. Then **113425** contains three occurrences of **1′1″2″** and seven occurrences of **1′-1″2″**

- **113425, 113425, 113425**
- 113425, 113425, 113425, 113425

- Avoidance of POPs $\leftrightarrow$ multi-avoidance of a set of patterns: avoiding 2′-1-2″ $\leftrightarrow$ simultaneously avoiding {2-1-2, 3-1-2, 2-1-3}.

Partially Ordered Patterns

Example

Let $\mathcal{T} = \{1', 1'', 2''\}$ with the only relation $1'' < 2''$. Then **113425** contains three occurrences of **1′1″2″** and seven occurrences of **1′-1″2″**

- **113425**, **113425**, **113425**
- **113425**, **113425**, **113425**, **113425**

- Avoidance of POPs ↔ multi-avoidance of a set of patterns: avoiding 2′-1-2″ ↔ simultaneously avoiding {2-1-2, 3-1-2, 2-1-3}.

Partially Ordered Patterns

Example

Let $\mathcal{T} = \{1', 1'', 2''\}$ with the only relation $1'' < 2''$. Then **113425** contains three occurrences of **1′1″2″** and seven occurrences of **1′-1″2″**

- **113425**, **113425**, **113425**
- **113425**, **113425**, **113425**, **113425**

- Avoidance of POPs $\leftrightarrow$ multi-avoidance of a set of patterns: avoiding 2′-1-2″ $\leftrightarrow$ simultaneously avoiding {2-1-2, 3-1-2, 2-1-3}.

One more Definition

Definition

A composition σ quasi-avoids a consecutive pattern τ if σ has exactly **one** occurrence of τ and the occurrence consists of the $|\tau|$ **rightmost parts** in σ.

Example

4112234 quasi-avoids **1123**
5223411 and **1123346** do **not** quasi-avoid **1123**

Some Notation

Generating functions

- $C_\tau^A(x) = \sum_{n\geq 0} |C_n^A(\tau)| x^n$
- $C_\tau^A(x;m) = \sum_{n\geq 0} |C_{n;m}^A(\tau)| x^n$
- $C_\tau^A(x,y) = \sum_{m\geq 0} C_\tau^A(x;m) y^m = \sum_{n,m\geq 0} |C_{n;m}^A(\tau)| x^n y^m$
- $D_\tau^A(x,y)$ = gf for the number of compositions in $C_{n;m}^A$ that quasi-avoid τ

Lemma

Let τ be a consecutive pattern. Then

$$D_\tau^A(x,y) = 1 + C_\tau^A(x,y)\left(y\sum_{a\in A} x^a - 1\right).$$

Proof: Adding the part a to the right of a composition with $m-1$ parts that avoids τ creates either a composition with m parts that still avoids τ or one that quasi-avoids τ. Thus, for $m \geq 1$,

$$\left(\sum_{a\in A} x^a\right) C_\tau^A(x;m-1) = C_\tau^A(x;m) + D_\tau^A(x;m).$$

■

Lemma

Let τ be a consecutive pattern. Then

$$D_\tau^A(x,y) = 1 + C_\tau^A(x,y)\left(y\sum_{a\in A} x^a - 1\right).$$

Proof: Adding the part a to the right of a composition with $m-1$ parts that avoids τ creates either a composition with m parts that still avoids τ or one that quasi-avoids τ. Thus, for $m \geq 1$,

$$\left(\sum_{a\in A} x^a\right) C_\tau^A(x;m-1) = C_\tau^A(x;m) + D_\tau^A(x;m).$$

■

Lemma

Let τ be a consecutive pattern. Then

$$D_\tau^A(x,y) = 1 + C_\tau^A(x,y)\left(y\sum_{a\in A} x^a - 1\right).$$

Proof: Adding the part a to the right of a composition with $m-1$ parts that avoids τ creates either a composition with m parts that still avoids τ or one that quasi-avoids τ. Thus, for $m \geq 1$,

$$\left(\sum_{a\in A} x^a\right) C_\tau^A(x;m-1) = C_\tau^A(x;m) + D_\tau^A(x;m).$$

■

Lemma

Let τ be a consecutive pattern. Then

$$D_\tau^A(x,y) = 1 + C_\tau^A(x,y)\left(y\sum_{a\in A} x^a - 1\right).$$

Proof: Adding the part a to the right of a composition with $m-1$ parts that avoids τ creates either a composition with m parts that still avoids τ or one that quasi-avoids τ. Thus, for $m \geq 1$,

$$\left(\sum_{a\in A} x^a\right) C_\tau^A(x;m-1) = C_\tau^A(x;m) + D_\tau^A(x;m).$$

■

Main Result

Theorem

Suppose $\tau = \tau_0$-ϕ, where ϕ is an arbitrary POP, and the letters of τ_0 are incomparable to the letters of ϕ. Then for all $k \geq 1$, we have

$$C_\tau^A(x,y) = C_{\tau_0}^A(x,y) + D_{\tau_0}^A(x,y) C_\phi^A(x,y).$$

Proof: Two possible cases:

- σ avoids $\tau_0 \Rightarrow C_{\tau_0}^A(x,y)$
- $\sigma = \sigma_1\sigma_2\sigma_3$ where σ_2 is the first occurrence of τ_0
  - $\sigma_1\sigma_2$ quasi-avoids the pattern τ_0
  - σ_3 must avoid ϕ

$\Rightarrow D_{\tau_0}^A(x,y) C_\phi^A(x,y)$

Main Result

Theorem

Suppose $\tau = \tau_0$-ϕ, where ϕ is an arbitrary POP, and the letters of τ_0 are incomparable to the letters of ϕ. Then for all $k \geq 1$, we have

$$C_\tau^A(x,y) = C_{\tau_0}^A(x,y) + D_{\tau_0}^A(x,y)C_\phi^A(x,y).$$

Proof: Two possible cases:

- σ avoids $\tau_0 \Rightarrow C_{\tau_0}^A(x,y)$
- $\sigma = \sigma_1\sigma_2\sigma_3$ where σ_2 is the first occurrence of τ_0
  - $\sigma_1\sigma_2$ quasi-avoids the pattern τ_0
  - σ_3 must avoid ϕ

$\Rightarrow D_{\tau_0}^A(x,y)C_\phi^A(x,y)$

Main Result

Theorem

Suppose $\tau = \tau_0$-ϕ, where ϕ is an arbitrary POP, and the letters of τ_0 are incomparable to the letters of ϕ. Then for all $k \geq 1$, we have

$$C_\tau^A(x,y) = C_{\tau_0}^A(x,y) + D_{\tau_0}^A(x,y) C_\phi^A(x,y).$$

Proof: Two possible cases:

- σ avoids $\tau_0 \Rightarrow C_{\tau_0}^A(x,y)$
- $\sigma = \sigma_1\sigma_2\sigma_3$ where σ_2 is the first occurrence of τ_0
  - $\sigma_1\sigma_2$ quasi-avoids the pattern τ_0
  - σ_3 must avoid ϕ

$\Rightarrow D_{\tau_0}^A(x,y) C_\phi^A(x,y)$ ■

Main Result

Theorem

Suppose $\tau = \tau_0$-ϕ, where ϕ is an arbitrary POP, and the letters of τ_0 are incomparable to the letters of ϕ. Then for all $k \geq 1$, we have

$$C_\tau^A(x,y) = C_{\tau_0}^A(x,y) + D_{\tau_0}^A(x,y)C_\phi^A(x,y).$$

Proof: Two possible cases:

- σ avoids $\tau_0 \Rightarrow C_{\tau_0}^A(x,y)$
- $\sigma = \sigma_1\sigma_2\sigma_3$ where σ_2 is the first occurrence of τ_0
  - $\sigma_1\sigma_2$ quasi-avoids the pattern τ_0
  - σ_3 must avoid ϕ

$\Rightarrow D_{\tau_0}^A(x,y)C_\phi^A(x,y)$ ∎

Main Result

Theorem

Suppose $\tau = \tau_0$-ϕ, where ϕ is an arbitrary POP, and the letters of τ_0 are incomparable to the letters of ϕ. Then for all $k \geq 1$, we have

$$C^A_\tau(x,y) = C^A_{\tau_0}(x,y) + D^A_{\tau_0}(x,y) C^A_\phi(x,y).$$

Proof: Two possible cases:

- σ avoids $\tau_0 \Rightarrow C^A_{\tau_0}(x,y)$
- $\sigma = \sigma_1\sigma_2\sigma_3$ where σ_2 is the first occurrence of τ_0
  - $\sigma_1\sigma_2$ quasi-avoids the pattern τ_0
  - σ_3 must avoid ϕ

$\Rightarrow D^A_{\tau_0}(x,y) C^A_\phi(x,y)$ ∎

Main Result

Theorem

Suppose $\tau = \tau_0$-ϕ, where ϕ is an arbitrary POP, and the letters of τ_0 are incomparable to the letters of ϕ. Then for all $k \geq 1$, we have

$$C^A_\tau(x,y) = C^A_{\tau_0}(x,y) + D^A_{\tau_0}(x,y)C^A_\phi(x,y).$$

Proof: Two possible cases:

- σ avoids $\tau_0 \Rightarrow C^A_{\tau_0}(x,y)$
- $\sigma = \sigma_1\sigma_2\sigma_3$ where σ_2 is the first occurrence of τ_0
    - $\sigma_1\sigma_2$ quasi-avoids the pattern τ_0
    - σ_3 must avoid ϕ

$\Rightarrow D^A_{\tau_0}(x,y)C^A_\phi(x,y)$ ∎

Equivalence of Patterns

Using equivalence of patterns, we will be able to establish results for families of patterns.

- Reversal map $R(\sigma) = R(\sigma_1\sigma_2\ldots\sigma_k) = \sigma_k\sigma_{k-1}\ldots\sigma_1$
- Reversal map R and identity map I are called trivial bijections of $C^A_{n;m}$ to itself
- τ_1 and τ_2 are equivalent, denoted by $\tau_1 \equiv \tau_2$, if $|C^A_{n;m}(\tau_1)| = |C^A_{n;m}(\tau_2)|$ for all A, m and n.
- $\tau \equiv R(\tau)$ for any pattern τ
- $\{\tau, R(\tau)\}$ = symmetry class of τ

Definition

Let $\{\tau_0, \tau_1, \ldots, \tau_s\}$ be a set of consecutive patterns.

- $\tau = \tau_1\text{-}\tau_2\text{-}\cdots\text{-}\tau_s$ is a **multi-pattern** if each letter of τ_i is incomparable with any letter of τ_j for $i \neq j$
- $\tau = \tau_0\text{-}a_1\text{-}\tau_1\text{-}a_2\text{-}\cdots\text{-}\tau_{s-1}\text{-}a_s\text{-}\tau_s$ is a **shuffle pattern** if each letter of τ_i is incomparable with any letter of τ_j for $i \neq j$ and the letters a_i are either all greater or all smaller than any letter of τ_j for any i and j.
- Shuffle pattern without the letters a_i $\rightarrow$ multi-pattern
- $1'\text{-}2\text{-}1''$ is a shuffle pattern, and $1'\text{-}1''$ is a multi-pattern.

Result for a Specific Multi-Pattern

Simplest non-trivial multi-pattern is $\Phi = 1' - 1''2''$. In this case we can derive the generating function directly:

- First letter can be any of the k letters in A
- All other letters have to be in non-increasing order

$$\begin{aligned} C^A_{1'\text{-}1''2''}(x,y) &= 1 + \left(y \sum_{a\in A} x^a\right) \prod_{a\in A} \left(\sum_{i\geq 0} (x^a y)^i\right) \\ &= 1 + \frac{y \sum_{a\in A} x^a}{\prod_{a\in A}(1-x^a y)}. \end{aligned}$$

Results for General Multi-Patterns

Theorem

Let $\tau = \tau_1\text{-}\tau_2\text{-}\cdots\text{-}\tau_s$ be a multi-pattern. Then

$$C_\tau^A(x,y) = \sum_{j=1}^{s} C_{\tau_j}^A(x,y) \prod_{i=1}^{j-1} \left[\left(y \sum_{a\in A} x^a - 1 \right) C_{\tau_i}^A(x,y) + 1 \right].$$

Proof: Follows from the lemma and the main result, together with induction. ∎

Results for General Multi-Patterns

Theorem

Let $\tau = \tau_1\text{-}\tau_2\text{-}\cdots\text{-}\tau_s$ be a multi-pattern. Then

$$C_\tau^A(x,y) = \sum_{j=1}^{s} C_{\tau_j}^A(x,y) \prod_{i=1}^{j-1} \left[\left(y \sum_{a\in A} x^a - 1 \right) C_{\tau_i}^A(x,y) + 1 \right].$$

Proof: Follows from the lemma and the main result, together with induction. ■

Results for Families of Multi-Patterns

Theorem

Let $\tau = \tau_0\text{-}\tau_1$ and $\phi = f_1(\tau_0)\text{-}f_2(\tau_1)$, where f_1 and f_2 are any of the trivial bijections. Then $\tau \equiv \phi$.

Proof: Claim: $\tau_0\text{-}\tau_1 \equiv \tau_0\text{-}f(\tau_1)$. If σ avoids $\tau_0\text{-}\tau_1$, then either

- σ has no occurrence of τ_0, so σ also avoids $\tau_0\text{-}f(\tau_1)$
- σ can be written as $\sigma = \sigma_1\sigma_2\sigma_3$, where $\sigma_1\sigma_2$ has exactly one occurrence of τ_0, namely σ_2. Then σ_3 must avoid τ_1, so $f(\sigma_3)$ avoids $f(\tau_1)$ and $\sigma_f = \sigma_1\sigma_2 f(\sigma_3)$ avoids $\tau_0\text{-}f(\tau_1)$.
- Converse also true $\Rightarrow$ bijection between class of compositions avoiding τ and those avoiding $\tau_0\text{-}f(\tau_1)$.
- This result and properties of trivial bijections finish proof.

Results for Families of Multi-Patterns

Theorem

Let $\tau = \tau_0\text{-}\tau_1$ and $\phi = f_1(\tau_0)\text{-}f_2(\tau_1)$, where f_1 and f_2 are any of the trivial bijections. Then $\tau \equiv \phi$.

Proof: Claim: $\tau_0\text{-}\tau_1 \equiv \tau_0\text{-}f(\tau_1)$. If σ avoids $\tau_0\text{-}\tau_1$, then either

- σ has no occurrence of τ_0, so σ also avoids $\tau_0\text{-}f(\tau_1)$
- σ can be written as $\sigma = \sigma_1\sigma_2\sigma_3$, where $\sigma_1\sigma_2$ has exactly one occurrence of τ_0, namely σ_2. Then σ_3 must avoid τ_1, so $f(\sigma_3)$ avoids $f(\tau_1)$ and $\sigma_f = \sigma_1\sigma_2 f(\sigma_3)$ avoids $\tau_0\text{-}f(\tau_1)$.
- Converse also true $\Rightarrow$ bijection between class of compositions avoiding τ and those avoiding $\tau_0\text{-}f(\tau_1)$.
- This result and properties of trivial bijections finish proof.

Results for Families of Multi-Patterns

Theorem

Suppose we have multi-patterns $\tau = \tau_1\text{-}\tau_2\text{-}\cdots\text{-}\tau_s$ *and* $\phi = \phi_1\text{-}\phi_2\text{-}\cdots\text{-}\phi_s$*, where* $\tau_1\tau_2\ldots\tau_s$ *is a permutation of* $\phi_1\phi_2\ldots\phi_s$*. Then* $\tau \equiv \phi$*.*

Proof: By induction. For $s = 2$, the previous theorem and properties of reversal maps give that

$$\tau_1\text{-}\tau_2 \equiv \tau_1\text{-}R(\tau_2) \equiv R(R(\tau_2))\text{-}R(\tau_1) \equiv \tau_2\text{-}R(R(\tau_1)) \equiv \tau_2\text{-}\tau_1.$$

General case follows with **careful** arguments and distinguishing two different cases. ∎

Results for Families of Multi-Patterns

Theorem

Suppose we have multi-patterns $\tau = \tau_1\text{-}\tau_2\text{-}\cdots\text{-}\tau_s$ *and* $\phi = \phi_1\text{-}\phi_2\text{-}\cdots\text{-}\phi_s$, *where* $\tau_1\tau_2\ldots\tau_s$ *is a permutation of* $\phi_1\phi_2\ldots\phi_s$. *Then* $\tau \equiv \phi$.

Proof: By induction. For $s = 2$, the previous theorem and properties of reversal maps give that

$$\tau_1\text{-}\tau_2 \equiv \tau_1\text{-}R(\tau_2) \equiv R(R(\tau_2))\text{-}R(\tau_1) \equiv \tau_2\text{-}R(R(\tau_1)) \equiv \tau_2\text{-}\tau_1.$$

General case follows with **careful** arguments and distinguishing two different cases. ∎

Results for Shuffle Patterns

Theorem

If ϕ be the shuffle pattern τ-ℓ-ν, then for all $k \geq \ell$,

$$C_{\phi}^{A}(x,y) = \frac{C_{\phi}^{A-\{a_k\}}(x,y) - x^{a_k} y C_{\tau}^{A-\{a_k\}}(x,y) C_{\nu}^{A-\{a_k\}}(x,y)}{(1 - x^{a_k} y C_{\tau}^{A-\{a_k\}}(x,y))(1 - x^{a_k} y C_{\nu}^{A-\{a_k\}}(x,y))}.$$

Note: For the shuffle pattern $\psi = \tau$-1-ν, replace a_k with a_1.

Results for Shuffle Patterns

Proof: Let $\phi = \tau\text{-}\ell\text{-}\nu$, $A' = A - \{a_k\}$, and assume σ contains exactly s copies of a_k.

- If $s = 0 \Rightarrow C_\phi^{A'}(x, y)$.
- If $s \geq 1$ then $\sigma = \sigma_0 a_k \sigma_1 a_k \cdots a_k \sigma_s$, where each σ_j is a ϕ-avoiding composition with parts in A'. Then either
  - σ_j avoids τ for all $j \Rightarrow x^{sa_k} y^s \left(C_\tau^{A'}(x, y)\right)^{s+1}$
  - $\exists\, j_0$ such that σ_{j_0} contains τ, σ_j avoids τ for all $j = 0, 1, \ldots, j_0 - 1$ and σ_j avoids ν for any $j = j_0 + 1, \ldots, s \Rightarrow$ $x^{sa_k} y^s \sum_{j=0}^{s} \left(C_\tau^{A'}(x, y)\right)^j \left(C_\nu^{A'}(x, y)\right)^{s-j} (C_\phi^{A'}(x, y) - C_\tau^{A'}(x, y))$
- Combine, simplify, use

$$\sum_{n\geq 0} x^n \sum_{j=0}^{n} p^j q^{n-j} = \frac{1}{(1 - xp)(1 - xq)}$$ to obtain result.

Results for Families of Shuffle Patterns

Corollary

Let $\phi = \tau\text{-}\ell\text{-}\nu$ *(resp.* $\phi = \tau\text{-}1\text{-}\nu$*) be a shuffle pattern, and let* $f(\phi) = f_1(\tau)\text{-}\ell\text{-}f_2(\nu)$ *(resp.* $f(\phi) = f_1(\tau)\text{-}1\text{-}f_2(\nu)$*), where* $f_1, f_2 \in \{R, I\}$ *are any trivial bijections. Then* $\phi \equiv f(\phi)$*.*

Corollary

For any shuffle pattern $\tau\text{-}\ell\text{-}\nu$ *(resp.* $\tau\text{-}1\text{-}\nu$*), we have* $\tau\text{-}\ell\text{-}\nu \equiv \nu\text{-}\ell\text{-}\tau$ *(resp.* $\tau\text{-}1\text{-}\nu \equiv \nu\text{-}1\text{-}\tau$*).*

Non-Overlapping Occurrences of POPs

- Two occurrences of a pattern τ overlap if they have any parts of σ in common
- τ-nlap(σ) = maximum number of non-overlapping occurrences of a consecutive pattern τ
- descent = 21 occurs at position *i* if $\sigma_i > \sigma_{i+1}$
- Two descents at positions *i* and *j* overlap if $j = i + 1$
- MND = maximum number of non-overlapping descents

  MND(333211) = 1

  MND(1332111143211) = 3

Non-Overlapping Occurrence of POPs

Theorem

Let τ be a consecutive pattern, τ-nlap(σ) is the maximum number of non-overlapping occurrences of τ in σ, and $g_\tau^A(x,y,t) = \sum_{n,m\geq 0}\sum_{\sigma\in C_{n;m}^A} x^n y^m t^{\tau\text{-nlap}(\sigma)}$. Then

$$g_\tau^A(x,y,t) = \frac{C_\tau^A(x,y)}{1-t\left[\left(y\sum_{a\in A}x^a-1\right)C_\tau^A(x,y)+1\right]}$$

Non-Overlapping Occurrence of POPs

Proof: Fix s and let $\Phi_s = \tau\text{-}\tau\text{-}\cdots\text{-}\tau$ with s copies of τ

- σ avoids $\Phi_s \Rightarrow \sigma$ has at most $s-1$ non-overlapping occurrences of τ
- Compute $C^A_{\Phi_{s+1}}(x,y)$ from general theorem for multi patterns
- Gf for number of compositions with exactly s non-overlapping copies of τ is given by $C^A_{\Phi_{s+1}}(x,y) - C^A_{\Phi_s}(x,y)$
- Sum over s

■

Non-Overlapping Occurrence of POPs

Example

- $C_{21}^{A}(x,y) = \prod_{a\in A} \frac{1}{(1-x^a y)}$
- Distribution of *MND* for the set $A = \{1,2\}$ is given by

$$\frac{1}{(1-x)(1-x^2)-x^3t} = \sum_{s\geq 0} \frac{x^{3s}}{(1-x)^{2s+2}(1+x)^{s+1}} t^s$$

- For $s = 2$, the sequence for the number of compositions for $n = 6, \ldots, 20$ is given by $\{$1, 3, 9, 19, 39, 69, 119, 189, 294, 434, 630, 882, 1218, 1638, 2178$\}$

Summary

- Gave recursive result for the gf for number of compositions that avoid a pattern of the form $\tau = \tau_0 - \Phi$
- Result applies directly to Multi-Patterns
- Result for Shuffle Patterns
- Application: gf for max number of non-overlapping occurrence of a POP in compositions

Preprint and this talk available from my web site at
sheubac@calstatela.edu

Preprint also at ArXiv
(http://www.arxiv.org/pdf/math.CO/0610030)

Article to appear in **Pu**re **M**athematics and **A**pplications

Thanks!