Hamiltonian Spectra of Trees

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Abstract

Let G be a connected graph, and let $d(u,v)$ denote the distance between vertices u and v in G. For any cyclic ordering π of $V(G)$, $\pi = (v_1, v_2, \cdots, v_n, v_{n+1} = v_1)$, let $\mathrm{d}(\pi) = \sum_{i=1}^{n} d(v_i, v_{i+1})$. The set of possible values of $\mathrm{d}(\pi)$ of all cyclic orderings π of $V(G)$ is called the *Hamiltonian spectrum* of G. We determine the Hamiltonian spectrum for any tree.

1 Introduction

Although not every connected graph is Hamiltonian (containing a spanning cycle), there always exists a closed spanning walk in a connected graph. A *Hamiltonian walk* of a connected graph is a shortest closed spanning walk; the length of such a walk is called the *Hamiltonian number* of G, denoted by $h(G)$. The value of $h(G)$ measures how far of G from being Hamiltonian. Let G be a connected graph on n vertices. Then $h(G) \geq n$, and the equality holds if and only if G is Hamiltonian.

A Hamiltonian walk can be expressed as a cyclic ordering of $V(G)$. Let $\pi = (v_1, v_2, \cdots, v_n, v_{n+1} = v_1)$ be a cyclic ordering of $V(G)$. Denote $\mathrm{d}(\pi)$ as the sum of the distances between pairs of consecutive vertices in π. That is,

$$\mathrm{d}(\pi) = \sum_{i=1}^{n} d(v_i, v_{i+1}).$$

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Let $\mathcal{H}(G)$ denote the set of values $\mathrm{d}(\pi)$ over all cyclic orderings π of $V(G)$. The minimum value in $\mathcal{H}(G)$ is indeed the Hamiltonian number [4], $h(G)$; the maximum value in $\mathcal{H}(G)$ is called the *upper Hamiltonian number*, denoted by $h^+(G)$.

The Hamiltonian number, the upper Hamiltonian number and the Hamiltonian spectrum for different families of graphs have been studied by several authors in the literature. It was proved by Goodman and Hedetniemi [6], and by Chartrand, Thomas and Zhang [4] (with a different approach) that for any connected graph G on n vertices, $h(G) \leq 2(n-1)$, and the equality holds if and only if G is a tree. Upper and lower bounds of $h^+(T)$ for a tree T were also given in [4]. The values of $h^+(G)$ for paths and odd cycles were obtained in [4, 5]. In [9], Král', Tong and Zhu determined the Hamiltonian spectrum for every cycle, and showed a sharp lower bound of $h^+(G)$ for general graphs G in terms of the order and the diameter of G.

Let G be a connected graph, and let v be a vertex of G. The *status* of v, denoted by $s(v)$, is the sum of distances from v to all other vertices (cf. Harary [7]). The minimum status over all vertices is called the *weight* of G, denoted by $W(G)$. A vertex with status $W(G)$ is called a *median*.

In this article, we use the median and weight to determine the Hamiltonian spectrum of a tree.

Theorem 1 *Let T be a tree on n vertices with weight $W(T)$. Then*

$$\mathcal{H}(T) = \{2(n-1), 2n, 2(n+1), \cdots, 2W(T)\}.$$

2 Proof of Theorem 1

Let T be a tree rooted at a vertex w. Define the *level function* on $V(T)$ by:

$$L_w(u) = d(w, u), \text{ for any } u \in V(T).$$

We shall simply use the notation $L(u)$ when the root w is understood in the context. Observe

Proposition 1 *Let T be a tree rooted at w. For any two vertices u and v, we have $d(u, v) \leq L_w(u) + L_w(v)$; and the equality holds if and only if u and v belong to different components of $T - w$, unless one of them is w.*

By definition, the status of a vertex w in a tree T has

$$s(w) = \sum_{u \in V(T)} L_w(u).$$

Hence, a median of T is a vertex w with the minimum $s(w)$ over all vertices.

For a vertex v in T, let $\kappa(v)$ be the maximum size of a component of $T - v$. A vertex w is a *centroid* if $\kappa(w)$ is the smallest among all vertices in T. Zelinka [11] proved that for a tree T, the median and centroid are identical. Hence, by results of Jordan [8] on centroid, the following results emerge.

Theorem A. *Let T be a tree on n vertices. Then w^* is a median of T if and only if each component of $T - w^*$ contains at most $n/2$ vertices.*

Theorem B. *Every tree T has at most two median vertices. If $|V(T)|$ is odd, then T has a unique median. If $|V(T)|$ is even, then T has two median vertices, say w and w', if and only if $ww' \in E(T)$ and the deletion of ww' from T results in two equal-sized components.*

Direct proofs of the above two theorems, without using centroid, can be found in [10], in which the weight of a tree is used to investigate multi-level distance labellings.

Proof of Theorem 1: Let T be a tree on n vertices. Observe that for every cyclic ordering π, $\mathrm{d}(\pi)$ is even. This is due to the facts that every edge in T is a cut-edge and $\mathrm{d}(\pi)$ is the length of a closed spanning walk, so each edge contributes an even number of times to the sum $\mathrm{d}(\pi) = \sum_{i=1}^{n} d(v_i, v_{i+1})$.

Next, we prove $h^+(T) = 2W(T)$. Let w be a median of T, $s(w) = W(T)$. Let $\pi = (v_1, v_2, \cdots, v_n)$ be a cyclic ordering of $V(T)$ with

$$h^+(T) = \mathrm{d}(\pi) = \sum_{i=1}^{n} d(v_i, v_{i+1}).$$

By Proposition 1 and since each vertex of T appears twice in the above summation, we have

$$h^+(T) \leq 2 \sum_{u \in V(T)} L_w(u) = 2W(T).$$

To show $h^+(T) = 2W(T)$, by Proposition 1, it suffices to find a cyclic ordering $\pi = (v_1, v_2, \cdots, v_n)$ such that the following is satisfied:

(*) For every i, v_i and v_{i+1} belong to different components of $T - w$, or one of v_i and v_{i+1} is w.

Let $F_1, F_2, \cdots, F_k$ be the components of $T - w$ with $|F_1| \geq |F_2| \geq \cdots \geq |F_k|$. By Theorem A, $|F_1| \leq n/2$. If $|F_1| = \lfloor n/2 \rfloor$, then we can find a cyclic ordering alternating between vertices in F_1 and vertices not in F_1, so **(*)** is satisfied. If $|F_1| = |F_k|$, then we can also easily get a cyclic ordering satisfying **(*)**, as $|F_1| = |F_2| = \cdots = |F_k|$. If $k = 2$, it is the case that $|F_1| = \lfloor n/2 \rfloor$.

Assume $k \geq 3$ and $|F_k| < |F_1| < \lfloor n/2 \rfloor$. We proceed by induction on n. By Theorems A and B, there exists an end-vertex $v \in F_k$ such that the tree $T' = T - v$ also has w as a median. By inductive hypothesis there exists a cyclic ordering π' of T' satisfying **(*)**. Since $|F_1| \geq (|F_k| - 1) + 2$ and $k \geq 3$, in π' there are two consecutive vertices v_i and v_{i+1} from two different components and $v_i, v_{i+1} \notin F_k$ (or one of v_i and v_{i+1} is w). The extension π of π' by inserting v between v_i and v_{i+1} is a cyclic ordering of T satisfying **(*)**.

It remains to show that for every $n \leq i \leq W(T)$, there is a cyclic ordering π such that $\mathrm{d}(\pi) = 2i$. We prove this by induction on n. It holds obviously when $n = 2$. Suppose it holds for trees with $n - 1$ vertices. By Theorems A and B, there exists an end-vertex v of T such that the tree $T' = T - v$ also has w as a median. Moreover, $W(T) = W(T') + d(w, v)$. Denote the path from w to v by $w = u_o, u_1, u_2, \cdots, u_t = v$, where $t = d(w, v)$. By inductive hypothesis, for each $n - 2 \leq i \leq W(T')$, there exists a cyclic ordering of $V(T')$ with $\mathrm{d}(\pi') = 2i$. Let π' be a cyclic ordering of $V(T')$. We extend π' to π by inserting v *after* u_{t-1}. That is

$$\pi' = (\cdots, u_{t-1}, y, \cdots) \;\Rightarrow\; \pi = (\cdots, u_{t-1}, v, y, \cdots).$$

It is easy to see that $\mathrm{d}(\pi) = \mathrm{d}(\pi') + 2$.

Let π' be a cyclic ordering of $V(T)$ with $\mathrm{d}(\pi') = 2W(T')$. To complete the proof of Theorem 1, it suffices to show that for every $2 \leq j \leq t$, π' can be extended to a cyclic ordering π of $V(T)$ with $\mathrm{d}(\pi) = \mathrm{d}(\pi') + 2j$. Since $\mathrm{d}(\pi') = 2W(T')$, we assume π' has the property **(*)**. Hence, we may assume in π' the vertex following v_{t-j}, say x, belongs to a different component (in $T' - w$) other than the one that v_{t-j} belongs to (or one of x and v_{t-j} is w). We insert v after v_{t-j} to get π. It is clear that $\mathrm{d}(\pi) = \mathrm{d}(\pi') + 2j$. This completes the proof of Theorem 1.

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