$\boxed{1.6}$

① (a) Prove by induction on n. If $n = 1$, then $\varphi(x) = \varphi(x)$. Assume $\varphi(x^k) = [\varphi(x)]^k$ for some integer k.

Then, $\varphi(x^{k+1}) = \varphi(x^k x) \underset{\uparrow}{=} \varphi(x^k)\varphi(x) =$

since φ is a hom

$\underset{\uparrow}{=} \varphi(x)^k \varphi(x) = [\varphi(x)]^{k+1}$.

by induction hypothesis

Lemma: $\varphi(1_G) = 1_H$. proof: $\varphi(1_G) = \varphi(1_G 1_G) = \varphi(1_G)\varphi(1_G)$. Thus, $\varphi(1_G) = 1_H$ ∎

Let 1_G and 1_H be the identities of G and H.

(b) $\varphi(x^{-1})\varphi(x) = \varphi(x^{-1}x) = \varphi(1_G) = 1_H$

$\varphi(x)\varphi(x^{-1}) = \varphi(xx^{-1}) = \varphi(1_G) = 1_H$

Hence $\varphi(x^{-1}) = \varphi(x)^{-1}$.

So, if $n < 0$, then

$$\varphi(x^n) = \varphi((x^{-1})^{-n}) = \varphi(x^{-1})^{-n}$$

$$= (\varphi(x)^{-1})^{-n} = \varphi(x)^n.$$

$\boxed{1.6}$

3 ($\Longrightarrow$) Suppose G is abelian. Let $x, y \in H$. Since φ is onto there exist $a, b \in G$ with $\varphi(a) = x$ and $\varphi(b) = y$. Since G is abelian $ab = ba$. Thus,

$$
\begin{aligned}
xy = \varphi(a)\varphi(b) &= \varphi(ab) = \varphi(ba) \\
&= \varphi(b)\varphi(a) = yx.
\end{aligned}
$$

So, H is abelian.

~~($\Longleftarrow$) Suppose H is abelian.~~

($\Longleftarrow$) <u>Lemma:</u> $\varphi^{-1} : H \to G$ is a homomorphirm. (We needed isomorphism so φ^{-1} exists)

<u>proof:</u> Let $a, b \in H$. Since φ is onto there exist $x, y \in G$ with $\varphi(x) = a$ and $\varphi(y) = b$. Since φ is a homomorphism $\varphi(xy) = \varphi(x)\varphi(y) = ab$. By definition of φ^{-1}, we have $\varphi^{-1}(x)\varphi^{-1}(y) = ab = \varphi^{-1}(xy)$. $\blacksquare$

Now apply the lemma to $\varphi^{-1} : H \to G$ to get that if H is abelian, then G is abelian.

1.6

(4) Suppose $\varphi : \mathbb{C}\setminus\{0\} \to \mathbb{R}\setminus\{0\}$ was an isomorphism. By the lemma for hw #1(b) solution, $\varphi(1) = 1$. Also, $\varphi(-1)^2 = \varphi((-1)^2) = \varphi(1) = 1$. Hence, $\varphi(-1) = \pm 1$ (since it solves $x^2 = 1$ where $x \in \mathbb{R}$). Since φ is 1-1, $\varphi(-1) = -1$.

Also, $\varphi(i)^4 = \varphi(i^4) = \varphi(1) = 1$. Thus, $\varphi(i) = \pm 1$. By φ is 1-1 so this is a contradiction. So, no such φ exists.

1.6

5 $\mathbb{R}$ is uncountable and $\mathbb{Q}$ is countable, hence there can be no 1-1 and onto function between them.

---

6 $\mathbb{Z}$ is cyclic. The following lemma will do the trick if you show that $G \simeq H$ isomorphic implies that G cyclic iff H cyclic.

Lemma: $\mathbb{Q}$ is not cyclic.

proof: Let $\frac{m}{n} \in \mathbb{Q}$ where $\frac{m}{n} \neq 0$. Then,

$$\left\langle \frac{m}{n} \right\rangle = \left\{ \ldots, \frac{-3m}{n}, \frac{-2m}{n}, \frac{-m}{n}, 0, \frac{m}{n}, \frac{2m}{n}, \frac{3m}{n}, \ldots \right\}.$$

Note that $\frac{m}{2n} \notin \left\langle \frac{m}{n} \right\rangle$ but $\frac{m}{2n} \in \mathbb{Q}$.

Hence $\mathbb{Q} \neq \left\langle \frac{m}{n} \right\rangle$. Thus, no element from $\mathbb{Q}$ generates all of $\mathbb{Q}$. $\blacksquare$

1.6

15 $\pi : \mathbb{R}^2 \to \mathbb{R}$

$\pi((x,y)) = x$

Let $(x,y), (a,b) \in \mathbb{R}^2$. Then

$$\pi\left[(x,y)+(a,b)\right] = \pi((x+a, y+b)) = x+a$$

$$= \pi((x,y)) + \pi((a,b)).$$

So, π is a hom.

$\ker(\pi) = \{(x,y) \mid \pi((x,y)) = 0\}$.

So, $\ker(\pi) = \{(0,y) \mid y \in \mathbb{R}\}$

which is the y-axis.