13.1.2: Show $\overbrace{x^3-2x-2}^{p(x)}$ is irreducible over $\mathbb{Q}$ and let θ be a root. Compute $(1+\theta)(1+\theta+\theta^2)$ and $\frac{1+\theta}{1+\theta+\theta^2}$ in $\mathbb{Q}(\theta)$

Proof: $p(x)$ is irreducible; applying Eisenstein's Criterion

$p=2$, $2\nmid 1$, $2 \mid -2$, $4 \nmid -2$

$\therefore$ $p(x)$ is irreducible over $\mathbb{Q}$ ✓

Note, $\theta^3-2\theta-2=0 \Rightarrow \theta^3 = 2\theta+2$

$$\Rightarrow (1+\theta)(1+\theta+\theta^2) = 1+\theta+\theta^2+\theta+\theta^2+\theta^3 =$$
$$= 1+2\theta+2\theta^2+2\theta+2 = 3+4\theta+2\theta^2.$$

$$\boxed{(1+\theta)(1+\theta+\theta^2) = 3+4\theta+2\theta^2}$$ ✓

Now we compute. $\frac{1+\theta}{1+\theta+\theta^2}$

$$\frac{1+\theta}{1+\theta+\theta^2} = a+b\theta+c\theta^2 \Leftrightarrow 1+\theta = (a+b\theta+c\theta^2)(1+\theta+\theta^2)$$
$$= a+a\theta+a\theta^2$$
$$+b\theta+b\theta^2+b\theta^3$$
$$+ \quad c\theta^2+c\theta^3+c\theta^4$$

$$\theta^4 = \theta^3\theta = (2\theta+2)\theta = 2\theta^2+2\theta$$

$$a+a\theta+a\theta^2$$
$$+b\theta+b\theta^2+b\theta^3 = b\theta+b\theta^2+b(2\theta+2)$$
$$= 2b+3b\theta+b\theta^2$$
$$c\theta^2+c\theta^3+c\theta^4 = c\theta^2+c(2\theta+2)+c(2\theta^2+2\theta)$$
$$= 3c\theta^2+4c\theta+2c$$

$\Longrightarrow$

$$a + a\theta + a\theta^2$$
$$+ 2b + 3b\theta + b\theta^2 \quad = 1 + \theta$$
$$+ 2c + 4c\theta + 3c\theta^2$$

$$\Longleftrightarrow \quad a + 2b + 2c - 1 = 0 \qquad (1)$$
$$\theta(a + 3b + 4c - 1) = 0 \qquad (2)$$
$$\theta^2(a + b + 3c) = 0 \qquad (3)$$

$$a = -b - 3c \overset{(1)}{\Longleftrightarrow} -b - 3c + 2b + 2c - 1 = 0$$
$$\Longleftrightarrow b - c - 1 = 0$$
$$b = c + 1$$

$$(2) \quad a + 3b + 4c - 1 = -b - 3c + 3b + 4c - 1 = 0$$
$$= 2b + c - 1 = 2(c+1) + c - 1$$
$$= 3c + 1 = 0$$
$$\boxed{c = -\frac{1}{3}}$$

$$b = -\frac{1}{3} + 1 = \boxed{\frac{2}{3} = b}$$

$$a = -\frac{2}{3} - 3\left(-\frac{1}{3}\right) = \frac{2}{3} + 1 = \boxed{\frac{5}{3} = a}$$

$$\therefore \frac{1+\theta}{1+\theta+\theta^2} = \frac{5}{3} + \frac{2}{3}\theta - \frac{1}{3}\theta^2 \in \mathbb{Q}[x]/(p(x))$$

$\blacksquare$