$\boxed{13.1}$

① $p(x) = x^3 + 9x + 6$ is irreducible using $q = 3$ and Eisenstein's criteria.

$$\mathbb{Q}(\theta) \cong \mathbb{Q}[x]/(x^3+9x+6)$$
$$\parallel$$
$$\{a + b\theta + c\theta^2 \mid \theta^3 + 9\theta + 6 = 0,\ a, b, c \in \mathbb{Q}\}$$

$\frac{1}{1+\theta} = a + b\theta + c\theta^2$ for some $a, b, c \in \mathbb{Q}$.

$$1 = a + b\theta + c\theta^2 + a\theta + b\theta^2 + c\theta^3$$
$$1 = a + b\theta + c\theta^2 + a\theta + b\theta^2 + c(-9\theta - 6)$$
$$1 = a + b\theta + c\theta^2 + a\theta + b\theta^2 + c\theta^2.$$
$$1 = (a - 6c) + (a + b - 9c)\theta + (b + c)\theta^2$$

So, $\left.\begin{array}{r} a - 6c = 1 \\ a + b - 9c = 0 \\ b + c = 0 \end{array}\right\}$ $\left(\begin{array}{ccc|c} 1 & 0 & -6 & 1 \\ 1 & 1 & -9 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & -6 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & 1 & 0 \end{array}\right)$

$\to \left(\begin{array}{ccc|c} 1 & 0 & -6 & +1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 4 & 1 \end{array}\right) \to \begin{array}{r} a - 6c = 1 \\ b - 3c = -1 \\ 4c = 1 \end{array}$

$\to c = \frac{1}{4} \to b = -1 + 3c = -1 + 3(\frac{1}{4}) = -1 + \frac{3}{4} = -\frac{1}{4}$

$a = 1 + 6c = 1 + \frac{6}{4} = \frac{10}{4} = \frac{5}{2}$

$$\boxed{\frac{1}{1+\theta} = \frac{5}{2} - \frac{1}{4}\theta + \frac{1}{4}\theta^2}$$

② $p(x) = x^3 - 2x - 2$ is irreducible using $q = 2$ and Eisenstein's criteria.

$\mathbb{Q}(\theta) = \{ a + b\theta + c\theta^2 \mid \theta^3 - 2\theta - 2 = 0,\ a,b,c \in \mathbb{Q} \}$.

$$(1+\theta)(1+\theta+\theta^2) = 1 + \theta + \theta^2 + \theta + \theta^2 + \theta^3$$
$$= 1 + 2\theta + 2\theta^2 + 2\theta + 2$$
$$= 3 + 4\theta + 2\theta^2$$

$$\frac{1+\theta}{1+\theta+\theta^2} = a + b\theta + c\theta^2$$

$$1+\theta = a + b\theta + c\theta^2 + a\theta + b\theta^2 + c\theta^3 + a\theta^2 + b\theta^3 + c\theta^4$$

$$1+\theta = a + (a+b)\theta + (a+b+c)\theta^2 + (a+c)\theta^3 + c\,\theta^4$$

$$1+\theta = a + (a+b)\theta + (a+b+c)\theta^2 + (a+c)(2\theta+2) + c(2\theta^2+2\theta)$$

$$1+\theta = (3a+2c) + (3a+b+4c)\theta + (a+b+3c)\theta^2$$

$$\begin{array}{r} {}^{2} \\ 29 \\ 3 \\ \hline 87 \end{array}$$

$$\left.\begin{aligned} 3a + 2c &= 1 \\ 3a + b + 4c &= 1 \\ a + b + 3c &= 0 \end{aligned}\right\} \left(\begin{array}{ccc|c} 3 & 0 & 2 & 1 \\ 3 & 1 & 4 & 1 \\ 1 & 1 & 3 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 3 & 0 & 2 & 1 \\ 3 & 1 & 4 & 1 \end{array}\right)$$

$$\xrightarrow[\substack{-3R_1+R_2 \to R_2 \\ -3R_1+R_3 \to R_3}]{} \left(\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & -3 & -7 & 1 \\ 0 & -2 & 5 & 1 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & 1 & 7/3 & -1/3 \\ 0 & -2 & 5 & 1 \end{array}\right) \xrightarrow[+2R_2+R_3 \to R_3]{} \left(\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & 1 & 7/3 & -1/3 \\ 0 & 0 & 29/3 & 1/3 \end{array}\right)$$

$$\left.\begin{aligned} a + b + 3c &= 0 \\ b + \tfrac{7}{3}c &= -1/3 \\ \tfrac{29}{3}c &= 1/3 \end{aligned}\right\} \to c = \frac{1}{29} \to b = -\frac{1}{3} - \frac{7}{3}\left(\frac{1}{29}\right) = -\frac{1}{3} - \frac{7}{87}$$

$$= \frac{-29-7}{87} = \frac{-36}{87}$$

$$a = -3c - b = -3\left(\frac{1}{29}\right) - \left(\frac{-36}{87}\right) = \frac{-9}{87} + \frac{36}{87} = \frac{27}{87}$$

$$\frac{1+\theta}{1+\theta+\theta^2} = \frac{27}{87} - \frac{36}{87}\theta + \frac{1}{29}\theta^2$$

③ $f(x) = x^3 + x + \bar{1}$ is irreducible over $\mathbb{Z}_2$ since $\deg(f) \leq 3$ and f has no roots in $\mathbb{Z}_2$ since $f(\bar{0}) = \bar{1}$, $f(\bar{1}) = \bar{1}$, $f(\bar{2}) = \bar{1}$.

$$\mathbb{F}_2(\theta) = \{ a + b\theta + c\theta^2 \mid \theta^3 = -\theta - \bar{1} = \theta + \bar{1},\ a, b, c \in \mathbb{Z}_2 \}$$

$$\theta^3 = \theta + \bar{1}$$
$$\theta^4 = \theta(\theta + \bar{1}) = \theta^2 + \theta$$
$$\theta^5 = \theta(\theta^2 + \theta) = \theta^3 + 2\theta = (\theta + 1) + 2\theta = \theta + \bar{1}$$

(now it starts repeating)
$$\theta^6 = \theta(\theta + \bar{1}) = \theta^2 + \theta$$
$$\theta^7 = \theta(\theta^2 + \theta) = \theta + \bar{1}$$
$$\vdots$$

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④ $\varphi(a + b\sqrt{2}) = a - b\sqrt{2}$

Let $a, b, c, d \in \mathbb{Q}$. Then

$$\varphi((a + b\sqrt{2}) + (c + d\sqrt{2})) = \varphi((a+c) + (b+d)\sqrt{2}) = (a+c) - (b+d)\sqrt{2}$$
$$= a - b\sqrt{2} + c - d\sqrt{2} = \varphi(a + b\sqrt{2}) + \varphi(c + d\sqrt{2})$$

$$\left.\begin{aligned} \varphi((a+b\sqrt{2})(c+d\sqrt{2})) &= \varphi(ac + 2bd + (ad+bc)\sqrt{2}) \\ &= ac + 2bd - (ad+bc)\sqrt{2} \\ \varphi(a+b\sqrt{2})\varphi(c+d\sqrt{2}) &= (a - b\sqrt{2})(c - d\sqrt{2}) \\ &= ac + 2bd - (ad+bc)\sqrt{2} \end{aligned}\right\} \underline{\text{equal}}.$$